SOCIETATEA DE ȘTIINȚE MATEMATICE DIN ROMÂNIA
ROMANIAN MATHEMATICAL SOCIETY

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Prove that if $a,b,c\in(0,\infty)$ ***then:*** $\Delta=\begin{vmatrix} s & \frac{a^2b}{a^3+b} & \frac{b^2c}{b^3+c} & \frac{c^2a}{c^3+a}\\ \frac{a^2b}{a^3+b} & s & \frac{c^2a}{c^3+a} & \frac{b^2c}{b^3+c}\\ \frac{b^2c}{b^3+c} & \frac{c^2a}{c^3+a} & s & \frac{a^2b}{a^3+b}\\ \frac{c^2a}{c^3+a} & \frac{b^2c}{b^3+c} & \frac{a^2b}{a^3+b} & s \end{vmatrix}>0$

$s-$ ***semiperimeter,*** a,b,c ***length sides in*** $\Delta\ ABC$

Proposed by Daniel Sitaru, Leonard Giugiuc – Romania

Solution 1 by Soumava Chakraborty – Kolkata – India

Solution 2 by authors

Solution 1 by Soumava Chakraborty – Kolkata – India

$$\Delta=s\begin{vmatrix} s & \frac{c^2a}{c^3+a} & \frac{b^2c}{b^3+c}\\ \frac{c^2a}{c^3+a} & s & \frac{a^2b}{a^3+b}\\ \frac{b^2c}{b^3+c} & \frac{a^2b}{a^3+b} & s \end{vmatrix}$$

$$-\frac{a^2b}{a^3+b}\begin{vmatrix} \frac{a^2b}{a^3+b} & \frac{c^2a}{c^3+a} & \frac{b^2c}{b^3+c}\\ \frac{b^2c}{b^3+c} & s & \frac{a^2b}{a^3+b}\\ \frac{c^2a}{c^3+a} & \frac{a^2b}{a^3+b} & s \end{vmatrix}+\frac{b^2c}{b^3+c}\begin{vmatrix} \frac{a^2b}{a^3+b} & s & \frac{b^2c}{b^3+c}\\ \frac{b^2c}{b^3+c} & \frac{c^2a}{c^3+a} & \frac{a^2b}{a^3+b}\\ \frac{c^2a}{c^3+a} & \frac{b^2c}{b^3+c} & s \end{vmatrix}$$

$$-\frac{c^2a}{c^3+a}\begin{vmatrix} \frac{a^2b}{a^3+b} & s & \frac{c^2a}{c^3+a} \\ \frac{b^2c}{b^3+c} & \frac{c^2a}{c^3+a} & s \\ \frac{c^2a}{c^3+a} & \frac{b^2c}{b^3+c} & \frac{a^2b}{a^3+b} \end{vmatrix}$$

$$\text{Let } \frac{a^2b}{a^3+b}=x, \frac{b^2c}{b^3+c}=y, \frac{c^2a}{c^3+a}=z$$

$$\Delta = s^2(s^2-x^2)+sz(xy-sz)+sy(zx-sy)$$
$$-x^2(s^2-x^2)-zx(zx-sy)-xy(xy-sz)$$
$$+xy(sz-xy)+sy(zx-sy)+y^2(y^2-z^2)$$
$$-zx(zx-sy)-sz(sz-xy)-z^2(y^2-z^2)$$
$$=(s^2-x^2)^2+(y^2-z^2)^2+2sz(xy-sz)-(xy-sz)(2xy)$$
$$+2sy(zx-sy)-2zx(zx-sy)$$
$$=(s^2-x^2)^2+(y^2-z^2)^2-2(xy-sz)^2-2(zx-sy)^2$$

Solution 2 by authors

$$a^3+b\ge 2a\sqrt{ab}\Rightarrow a^3-2a\sqrt{ab}+b\ge 0\Rightarrow \left(a\sqrt{a}-\sqrt{b}\right)^2\ge 0$$

$$\frac{a^2b}{a^3b}\le\frac{a^2b}{2a\sqrt{ab}}=\frac{ab}{2\sqrt{ab}}=\frac{\sqrt{ab}}{2}\le\frac{1}{2}\cdot\frac{a+b}{2}=\frac{a+b}{4}$$

Analogous:

$$\frac{b^2c}{b^3+c}\le\frac{b+c}{4};\frac{c^2a}{c^3+a}\le\frac{c+a}{4}$$

By adding:

$$\sum\frac{a^2b}{a^3+b}\le\frac{a+b+c}{2}=s$$

$$\Rightarrow p \geq \sum \frac{a^2 b}{a^3+b} \quad (1)$$

Lemma:

If $x, y, z, t \in \mathbb{R}$ ***then:***

$$\Delta' = \begin{vmatrix} x & y & z & t \\ y & x & t & z \\ z & t & x & y \\ t & z & y & x \end{vmatrix} = (x+y+z+t)(x-y+z-t)(x+y-z-t)(x-y-z+t)$$

$$\begin{vmatrix} x & y & z & t \\ y & x & t & z \\ z & t & x & y \\ t & z & y & x \end{vmatrix}^{L_1+L_2+L_3+L_4} = (x+y+z+t)\begin{vmatrix} 1 & 1 & 1 & 1 \\ y & x & t & z \\ z & t & x & y \\ t & z & y & x \end{vmatrix}^{C_4-C_3+C_2-C_1} =$$

$$= (x+y+z+t)\begin{vmatrix} 1 & 1 & 1 & 0 \\ y & x & t & z-t+x-y \\ z & t & x & -(z-t+x-y) \\ t & z & y & z-t+x-y \end{vmatrix} =$$

$$= (x+y+z+t)(x-y+z-t)\begin{vmatrix} 1 & 1 & 1 & 0 \\ y & x & t & 1 \\ z & t & x & -1 \\ t & z & y & 1 \end{vmatrix}^{\substack{C_2-C_1 \\ C_3-C_1}} =$$

$$= (x+y+z+t)(x-y+z-t)\begin{vmatrix} 1 & 0 & 0 & 0 \\ y & x-y & t-z & 1 \\ z & t-z & x-z & -1 \\ t & z-t & y-t & 1 \end{vmatrix} =$$

$$= (x+y+z+t)(x-y+z-t)\begin{vmatrix} x-y & t-z & 1 \\ t-z & x-z & -1 \\ z-t & y-t & 1 \end{vmatrix}^{\substack{L_1+L_2 \\ L_3+L_2}} =$$

$$= (x+y+z+t)(x-y+z-t)\begin{vmatrix} x-y+t-z & x-y-z+t \\ 0 & x-z+y-t \end{vmatrix} =$$

$$= (x+y+z+t)(x-y+z-t)(x-y+t-x)(x+y-z-t)$$

If $x, y, z, t > 0; x > y+z+t$ ***then*** $\Delta' > 0$

$$x-y+z-t > y+z+t-y+z-t = 2z > 0$$

$$x+y-z-t > y+z+t+y-z-t = 2y > 0$$

$$x-y-z+t > y+z+t-y-z+t = 2t > 0$$

Let be $x = \frac{a^2b}{a^3+b}; y = \frac{b^2c}{b^3+c}; z = \frac{c^2a}{c^3+a}$

From (1) and (2) it follows $\Delta = \Delta' > 0$