In $ABCD$ convexe quadrilateral: $AB=a, BC=b, CD=c, DA=d$. Prove that:

$$\sum\sqrt{a^2+b^2+c^2}>2\sqrt{3\cdot AC\cdot BD}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by SK Rejuan-West Bengal-India
Solution 2 by Myagmarsuren Yadamsuren – Mongolia

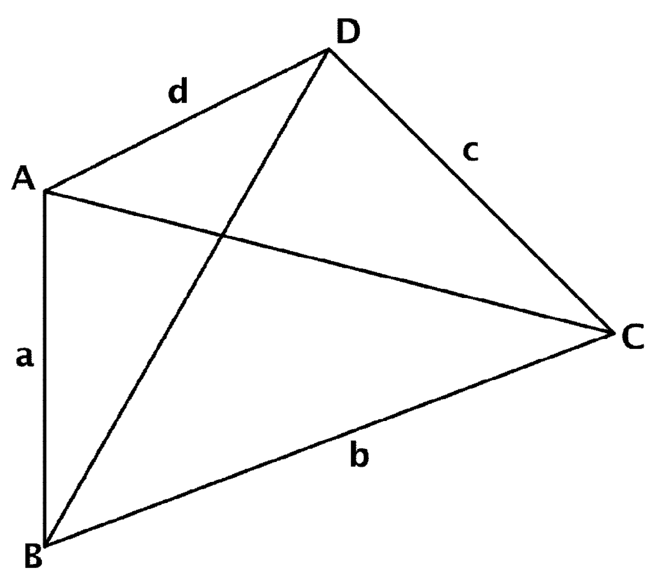
Solution 1 by SK Rejuan-West Bengal-India

$$(a^2+b^2+c^2)>\frac{1}{3}(a+b+c)^2$$ **[by mth power theorem]**

$$\Rightarrow\sqrt{a^2+b^2+c^2}>\frac{1}{\sqrt{3}}(a+b+c)$$

$$\Rightarrow\sum\sqrt{a^2+b^2+c^2}>\frac{1}{\sqrt{3}}\sum(a+b+c)=\frac{3}{\sqrt{3}}(a+b+c+d)$$

$$\Rightarrow\sum\sqrt{a^2+b^2+c^2}>\sqrt{3}(a+b+c)\quad \textbf{(1)}$$



For, $\Delta\ \mathbf{ABC}, \mathbf{a} + \mathbf{b} > \mathbf{AC}$
$\Delta\mathbf{BCD}, \mathbf{b} + \mathbf{c} > \mathbf{BD}$
$\Delta\ \mathbf{CDA}, \mathbf{c} + \mathbf{d} > \mathbf{AC}$
$\Delta\mathbf{DAB}, \mathbf{d} + \mathbf{a} > \mathbf{BD}$

Adding the we get, $2(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}) > 2(\mathbf{AC} + \mathbf{BD})$

$$\Rightarrow \sqrt{3}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}) > \sqrt{3}(\mathbf{AC} + \mathbf{BD}) \quad \textbf{(2)}$$

Also by $\mathbf{AM} > \mathbf{GM}$ **we get,**

$$\mathbf{AC} + \mathbf{BD} > 2\sqrt{\mathbf{AC} \cdot \mathbf{BD}}$$

$$\Rightarrow \sqrt{3}(\mathbf{AC} + \mathbf{BD}) > 2\sqrt{3 \cdot \mathbf{AC} \cdot \mathbf{BD}} \quad \textbf{(3)}$$

From (1), (2) & (3) we get,

$$\sum \sqrt{\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2} > \sqrt{3}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}) > \sqrt{3}(\mathbf{AC} + \mathbf{BD}) > 2\sqrt{3 \cdot \mathbf{AC} \cdot \mathbf{BD}}$$

$$\Rightarrow \sum \sqrt{\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2} > 2\sqrt{3 \cdot \mathbf{AC} \cdot \mathbf{BD}}$$

Solution 2 by Myagmarsuren Yadamsuren – Mongolia

In $\boldsymbol{ABCD}$ ***– CONVEXE quadrilater:***

$\boldsymbol{AB = a; BC = b; CD = c; DA = d}$***. Prove that***

$$\sum \sqrt{\boldsymbol{a^2 + b^2 + c^2}} > 2 \cdot \sqrt{3 \cdot \boldsymbol{AC} \cdot \boldsymbol{BD}}$$

$$\sqrt{\boldsymbol{a^2 + b^2 + c^2}} + \sqrt{\boldsymbol{b^2 + c^2 + d^2}} + \sqrt{\boldsymbol{c^2 + d^2 + a^2}} + \sqrt{\boldsymbol{d^2 + a^2 + b^2}} \geq$$
$$\geq \sqrt{3 \cdot (\boldsymbol{a^2 + b^2 + c^2 + d^2})}$$

$$\Rightarrow \sqrt{\boldsymbol{a^2 + b^2 + c^2}} \cdot \sqrt{\boldsymbol{b^2 + c^2 + d^2}} > 0 \textit{ (apparently inequality)}$$

$$(a^2+b^2+c^2)+2\cdot\sqrt{a^2+b^2+c^2}\cdot\sqrt{b^2+c^2+d^2}+(b^2+c^2+d^2)>$$
$$>(a^2+2b^2+2c^2+d^2)\Rightarrow$$
$$\Rightarrow\left(\sqrt{a^2+b^2+c^2}+\sqrt{b^2+c^2+d^2}\right)^2>(a^2+2b^2+2c^2+d^2)$$

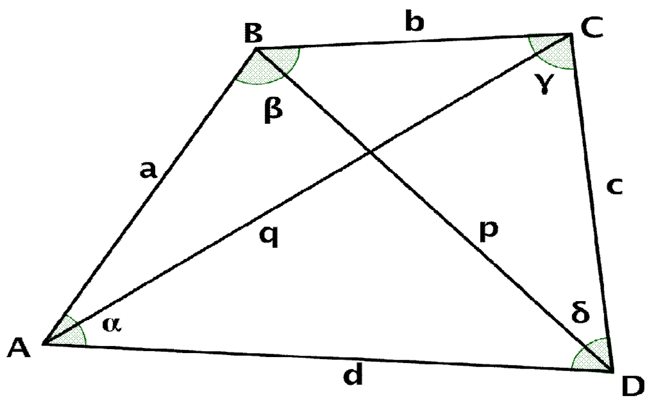
$$\begin{cases}\sqrt{a^2+b^2+c^2}+\sqrt{b^2+c^2+d^2}>\sqrt{a^2+2b^2+2c^2+d^2}\\ \sqrt{c^2+d^2+a^2}+\sqrt{a^2+b^2+c^2}>\sqrt{2a^2+b^2+c^2+2d^2}\end{cases}$$

similarly

$$\sqrt{a^2+2b^2+2c^2+d^2}+\sqrt{2a^2+b^2+c^2+2d^2}>\sqrt{3(a^2+b^2+c^2+d^2)}$$

$$\sum\sqrt{a^2+b^2+c^2}>\sqrt{3(a^2+b^2+c^2+d^2)}$$

To assure: $\sqrt{3\cdot(a^2+b^2+c^2+d^2)}>2\cdot\sqrt{3\cdot AC\cdot BD}$ ***(*)***



$$AB=a,BC=b,CD=c,DA=d,AC=q,BD=p$$

$\cos(t_2)$:

$$\sum\begin{matrix}p^2=a^2+d^2-2ad\cdot\cos\alpha\\ p^2=b^2+c^2-2bc\cdot\cos\gamma\\ q^2=a^2+b^2-2ab\cdot\cos\beta\\ q^2=c^2+d^2-2cd\cdot\cos\delta\end{matrix}$$

$$2\cdot(p^2+q^2)=2\cdot(a^2+b^2+c^2+d^2)-$$
$$-2\cdot(ad\cdot\cos\alpha+ab\cdot\cos\beta+bc\cdot\cos\gamma+cd\cdot\cos\delta)$$

$$p^2+q^2=a^2+b^2+c^2+d^2-(ad\cdot\cos\alpha+ab\cdot\cos\beta+bc\cdot\cos\gamma+ca\cdot\cos\delta)$$

$$a^2+b^2+c^2+d^2=p^2+q^2+(ad\cdot\cos\alpha+ab\cdot\cos\beta+bc\cdot\cos\gamma+cd\cdot\cos\delta)\overset{CBC}{\geq}$$

$$\geq p^2+q^2+\frac{1}{4}(ab+bc+cd+da)\cdot(\cos\alpha+\cos\beta+\cos\gamma+\cos\delta)$$

Let's: $\left.\begin{matrix}\beta\geq\alpha\geq\gamma\geq\delta\\ \alpha+\gamma=180^\circ\\ \beta+\delta=180^\circ\end{matrix}\right\}\Rightarrow\cos\alpha+\cos\beta+\cos\gamma+\cos\delta\geq$

$$\geq 2\cdot(\cos\alpha+\cos\gamma)=4\cdot\cos\frac{\alpha+\gamma}{2}\cdot\cos\frac{\alpha-\gamma}{2}=$$
$$=4\cdot\cos 90^\circ\cdot\cos\frac{\alpha-\gamma}{2}=0$$

There:

$$a^2+b^2+c^2+d^2=p^2+q^2+x^2$$

$$a^2+b^2+c^2+d^2>p^2+q^2$$

(*) $\Rightarrow\sqrt{3\cdot(a^2+b^2+c^2+d^2)}>\sqrt{3\cdot(p^2+q^2)}\overset{Cauchy}{\geq}2\cdot\sqrt{3pq}$