DINCA’S REFINEMENT FOR NESBITT’S INEQUALITY

If \(a, b, c > 0\) then:

\[
\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \geq \frac{(a + b + c)^2}{2(ab + bc + ca)} \geq \frac{3\sqrt[3]{3(a^2 + b^2 + c^2)}}{2(a + b + c)} \geq \frac{3}{2}
\]

Proof 1 by Leonard Giugiuc – Romania
Proof 2 by Soumitra Moukherjee – Chandar Nagore – India

Proof 1 by Leonard Giugiuc – Romania

Let WLOG \(a + b + c = 3, ab + bc + ca = 3(1 - t^2), 0 \leq t < 1\) and \(abc = p\).

After easy algebraic manipulations, the first part is equivalent to

\[
\frac{3 + 6t^2 + p}{9(1-t^2) - p} \geq \frac{1}{2(1-t^2)}; \text{ for every fixed in } [0, 1) \text{ consider the function}
\]

\[
f_t(x) = \frac{3 + 6t^2 + x}{9(1-t^2) - x} \text{ on the domain } [0, \max p]. \text{ Clearly, } f_t \text{ is increasing. Also, it’s well known that}
\]

\[
\min p = \begin{cases} 
(1 + t)^2(1 - 2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\
0, & \text{if } \frac{1}{2} \leq t < 1
\end{cases}
\]

Case 1: \(0 \leq t \leq \frac{1}{2}\) \(\Rightarrow f_t(p) \geq f_t ((1 + t)^2(1 - 2t))\). Hence it suffices to prove

\[
f_t ((1 + t)^2(1 - 2t)) \geq \frac{1}{2(1-t)^2} \iff \frac{-2t^3 + 3t^2 + 4}{(2-t)^2} \geq \frac{1}{1-t} \iff t^2(1 - 2t)(2 - t) \geq 0,
\]

which is obvious.
Note that equality holds for \( t \in \left\{ 0, \frac{1}{2} \right\} \) i.e at \((a, a, a)\) or at \((a, a, 0)\) + permutations, \( a > 0 \).

Case 2: \( \frac{1}{2} \leq t < 1 \). Then \( f_t(p) \geq f_t(0) = \frac{1+2t^2}{3(1-t^2)} \). Hence it suffices to prove

\[
\frac{1+2t^2}{3(1-t^2)} \geq \frac{1}{2(1-t^2)} \iff 4t^2 \geq 1, \text{ which is true.}
\]

The second part is equivalent to \((1 + 2t^2)(1 - t^2)^2 \leq 1\), which is obvious.

Equality at \((a, a, a)\).

This completes the proof.

Proof 2 by Soumitra Moukherjee - Chandar Nagore - India

\[
\sum_{\text{cyc}} \frac{a}{b+c} = \sum_{\text{cyc}} \frac{a^2}{ab+ac} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)} \quad \text{[Applying Bergstrom's Ineq]}
\]

we will prove, \( \frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{3\sqrt{3\sum_{\text{cyc}} a^2}}{2(a+b+c)} \)

\[
\implies \frac{p^2}{q} \geq \frac{3\sqrt{3(p^2 - 2q^2)}}{p} \quad \text{where } p = a + b + c \text{ and } q = ab + bc + ca
\]

\[
\implies p^6 \geq 27q^2(p^2 - 2q) \implies p^6 - 27q^3 - 27q^2(p^2 - 3q) \geq 0
\]

\[
\implies (p^2 - 3q)(p^4 + 3p^2q - 18q^2) \geq 0
\]

\[
\implies (p^2 - 3q)^2(p^2 + 6q) \geq 0, \text{ which is true again,}
\]
\[ 3 \left( \sum_{\text{cyc}} a^2 \right) \geq \left( \sum_{\text{cyc}} a \right)^2 \Rightarrow \frac{3 \sqrt{3 \left( \sum_{\text{cyc}} a^2 \right)}}{2 \left( \sum_{\text{cyc}} a \right)} \geq \frac{3}{2} \]

so,

\[ \sum_{\text{cyc}} \frac{a}{b+c} \geq \frac{\left( \sum_{\text{cyc}} a \right)^2}{2 \left( \sum_{\text{cyc}} ab \right)} \geq \frac{3 \sqrt{3 \left( \sum_{\text{cyc}} a^2 \right)}}{2 \left( \sum_{\text{cyc}} a \right)} \geq \frac{3}{2} \]

(proved)