SOCIETATEA DE ȘTIINȚE MATEMATICE DIN ROMÂNIA
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DINCA'S REFINEMENT FOR NESBITT'S INEQUALITY

If $a, b, c > 0$ then:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{3\sqrt{3(a^2+b^2+c^2)}}{2(a+b+c)} \geq \frac{3}{2}$$

Proof 1 by Leonard Giugiuc – Romania
Proof 2 by Soumitra Moukherjee - Chandar Nagore – India

Proof 1 by Leonard Giugiuc – Romania

Let WLOG $a+b+c=3, ab+bc+ca=3(1-t^2), 0 \leq t < 1$ and $abc = p$.

After easy algebraic manipulations, the first part is equivalent to

$\frac{3+6t^2+p}{9(1-t^2)-p} \geq \frac{1}{2(1-t^2)}$ **; for every fixed in $[0,1)$ consider the function**

$f_t(x) = \frac{3+6t^2+x}{9(1-t^2)-x}$ **on the domain $[0, \max p]$. Clearly, f_t is increasing. Also, it's well known that**

$$\min p = \begin{cases} (1+t)^2(1-2t), \text{ if } 0 \leq t \leq \frac{1}{2} \\ 0, \text{ if } \frac{1}{2} \leq t < 1 \end{cases}$$

Case 1: $0 \leq t \leq \frac{1}{2} \Rightarrow f_t(p) \geq f_t\left((1+t)^2(1-2t)\right)$. Hence it suffices to prove

$$f_t\left((1+t)^2(1-2t)\right) \geq \frac{1}{2(1-t)^2} \Leftrightarrow \frac{-2t^3+3t^2+4}{(2-t)^2} \geq \frac{1}{1-t} \Leftrightarrow t^2(1-2t)(2-t) \geq 0,$$

which is obvious.

Note that equality holds for $t \in \left\{0, \frac{1}{2}\right\}$ i.e at (a, a, a) or at $(a, a, 0)$ + permutations, $a > 0$.

Case 2: $\frac{1}{2} \leq t < 1$. Then $f_t(p) \geq f_t(0) = \frac{1+2t^2}{3(1-t^2)}$. Hence it suffices to prove

$$\frac{1+2t^2}{3(1-t^2)} \geq \frac{1}{2(1-t^2)} \Leftrightarrow 4t^2 \geq 1, \text{ which is true.}$$

The second part is equivalent to $(1 + 2t^2)(1 - t^2)^2 \leq 1$, which is obvious.

Equality at (a, a, a).

This completes the proof.

Proof 2 by Soumitra Moukherjee - Chandar Nagore – India

$$\sum_{cyc} \frac{a}{b+c} = \sum_{cyc} \frac{a^2}{ab+ac} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)} \text{ [Applying Bergstrom's Ineq]}$$

we will prove, $\frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{3\sqrt{3(\sum_{cyc} a^2)}}{2(a+b+c)}$

$$\Rightarrow \frac{p^2}{q} \geq \frac{3\sqrt{3(p^2-2q)}}{p} \text{ where } p = a + b + c \text{ and } q = ab + bc + ca$$

$$\Rightarrow p^6 \geq 27q^2(p^2 - 2q) \Rightarrow p^6 - 27q^3 - 27q^2(p^2 - 3q) \geq 0$$

$$\Rightarrow (p^2 - 3q)(p^4 + 3p^2q - 18q^2) \geq 0$$

$$\Rightarrow (p^2 - 3q)^2(p^2 + 6q) \geq 0, \text{ which is true}$$

again,

$$3\left(\sum_{cyc} a^2\right) \geq \left(\sum_{cyc} a\right)^2 \Rightarrow \frac{3\sqrt{3\left(\sum_{cyc} a^2\right)}}{2\left(\sum_{cyc} a\right)} \geq \frac{3}{2}$$

so,

$$\sum_{cyc} \frac{a}{b+c} \geq \frac{\left(\sum_{cyc} a\right)^2}{2\left(\sum_{cyc} ab\right)} \geq \frac{3\sqrt{3\left(\sum_{cyc} a^2\right)}}{2\left(\sum_{cyc} a\right)} \geq \frac{3}{2}$$

(proved)