JP.026. Let a, b, c be non-negative real numbers and let x, y, z be real numbers different from 0, such that $by + cz = x$, $cz + ax = y$, $ax + by = z$. Prove that.

a. $abc \leq \frac{1}{8}$

b. $\frac{1}{2+a+b} + \frac{1}{2+b+c} + \frac{1}{2+c+a} \leq 1$

c. $a + b + c \geq 2(ab + bc + ca)$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Sean: a, b, c números reales no negativos y sean: x, y, z números reales diferentes de cero.
Además: $by + cz = x, cz + ax = y$, $ax + by = z$. Probar que:

a) $abc \leq \frac{1}{8}$

b) $\frac{1}{2+a+b} + \frac{1}{2+b+c} + \frac{1}{2+a+c} \leq 1$

c) $a + b + c \geq 2(ab + bc + ac)$

Sumando las ecuaciones se tiene:

$$2(ax + by + cz) = x + y + z \rightarrow (ax + by) + cz = \frac{x + y + z}{2} \rightarrow$$

$$\rightarrow cz = \frac{x + y - z}{2} \rightarrow c = \frac{x + y - z}{2z}$$

De forma análoga se tiene que:

$$a = \frac{y + z - x}{2x}$$

$$b = \frac{z + x - y}{2y}$$

Por la tanto:

a) $abc \le \frac{1}{8} \Leftrightarrow (x + y - z)(z + x - y)(y + z - x) \le xyz$

$$(x + y - z)(z + x - y)(y + z - x) = (x^2 - (y - z)^2)(y + z - x) =$$
$$= (x^2 - y^2 - z^2 + 2yz)(y + z - x)$$

$$\to (x^2 - y^2 - z^2 + 2yz)(y + z - x) = x^2y + x^2z - x^3 - y^3 - y^2z + y^2x -$$
$$-z^2y - z^3 + z^2x + 2y^2z + 2yz^2 - 2xyz$$

$$\Rightarrow (x + y - z)(z + x - y)(y + z - x) = -x^3 - y^3 - z^3 + xy(x + y) +$$
$$+yz(y + z) + zx(z + x) - 2xyz \le xyz$$

$$\Rightarrow x^3 + y^3 + z^3 + 3xyz \ge xy(x + y) + yz(y + z) + zx(z + x)$$

$$\Rightarrow x(x - y)(x - z) + y(y - x)(y - z) + z(z - x)(z - y) \ge 0 \to$$

→ (Válido por desigualdad de Schur)

b) $\frac{1}{2+a+b} + \frac{1}{2+b+c} + \frac{1}{2+a+c} \le 1$

$$\to (2 + a + c)(2 + b + c) + (2 + a + b)(2 + a + c) + (2 + a + b)(2 + b + c) \le$$
$$\le (2 + a + b)(2 + b + c)(2 + a + c)$$

$$\to 4(3) + \sum (a + c)(b + c) + 2\sum (2a + b + c) \le 8 + 4(2)(a + b + c) +$$
$$+2\sum (a + c)(b + c) + \prod (a + b)$$
$$\to 4 + 8(a + b + c)$$
$$\le 8(a + b + c) + \sum (a + c)(b + c) + \sum ab(a + b) + 2abc$$

$$\Rightarrow 4 \le \sum a^2 + 3\sum ab + \sum ab(a+b) + 2abc$$

Tener en cuenta lo siguiente:

$$a = \frac{y+z-x}{2x} \ge 0$$
$$b = \frac{z+x-y}{2y} \ge 0$$
$$c = \frac{x+y-z}{2z} \ge 0$$

$$\Rightarrow \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = 2 \rightarrow \sum (1+b)(1+c) = 2\prod (1+a) \rightarrow$$

$$\rightarrow 3 + 2\sum a + \sum ab = 2 + 2\sum a + 2\sum ab + 2abc$$

$$\Rightarrow 1 = ab + bc + ac + 2abc \rightarrow ab + bc + ac = 1 - 2abc \ \wedge abc \le \frac{1}{8} \Leftrightarrow$$

$$\Leftrightarrow ab + bc + ac \ge \frac{3}{4}$$

$$\Rightarrow a + b + c \ge \sqrt{3(ab + bc + ac)} \ge \sqrt{\frac{9}{4}} = \frac{3}{2} \rightarrow a + b + c \ge \frac{3}{2}$$

$$\sum a^2 + 3\sum ab + \sum ab(a+b) + 2abc \ge 4\sum ab + \sum ab\left(\frac{3}{2} - c\right) + 2abc$$

$$= \frac{11}{2}\sum ab - abc \ge \frac{33}{8} - \frac{1}{8} \ge 4$$

c) $a+b+c\geq 2(ab+bc+ca)$

Desde que:

$$a=\frac{y+z-x}{2x}\geq 0$$

$$b=\frac{z+x-y}{2y}\geq 0$$

$$c=\frac{x+y-z}{2z}\geq 0$$

$$\Leftrightarrow y=n+m;z=n+p;x=m+p$$

$$a=\frac{n}{m+p}\geq 0$$

$$b=\frac{p}{n+m}\geq 0$$

$$c=\frac{m}{n+p}\geq 0$$

$$\Rightarrow \frac{n}{m+p}+\frac{p}{m+n}+\frac{m}{n+p}\geq 2\left(\frac{n}{m+p}\cdot\frac{p}{m+n}+\frac{n}{m+p}\cdot\frac{m}{n+p}+\frac{p}{m+n}\cdot\frac{m}{n+p}\right)$$

$$\Rightarrow \frac{n(n+m)(n+p)+p(p+m)(p+n)+m(m+p)(m+n)}{(m+n)(n+p)(m+p)}\geq$$

$$\geq\frac{2np(n+p)+mn(m+n)+mp(m+p)}{(m+n)(n+p)(m+p)}$$

$$\Rightarrow \sum n^3-\sum mn(m+n)+3mnp\geq 0$$

$$\Rightarrow \boldsymbol{n(n-m)(n-p)} + \boldsymbol{m(m-n)(m-p)} + \boldsymbol{p(p-m)(p-n)} \geq \boldsymbol{0} \Leftrightarrow$$

⇔ **(Desigualdad de Schur)**