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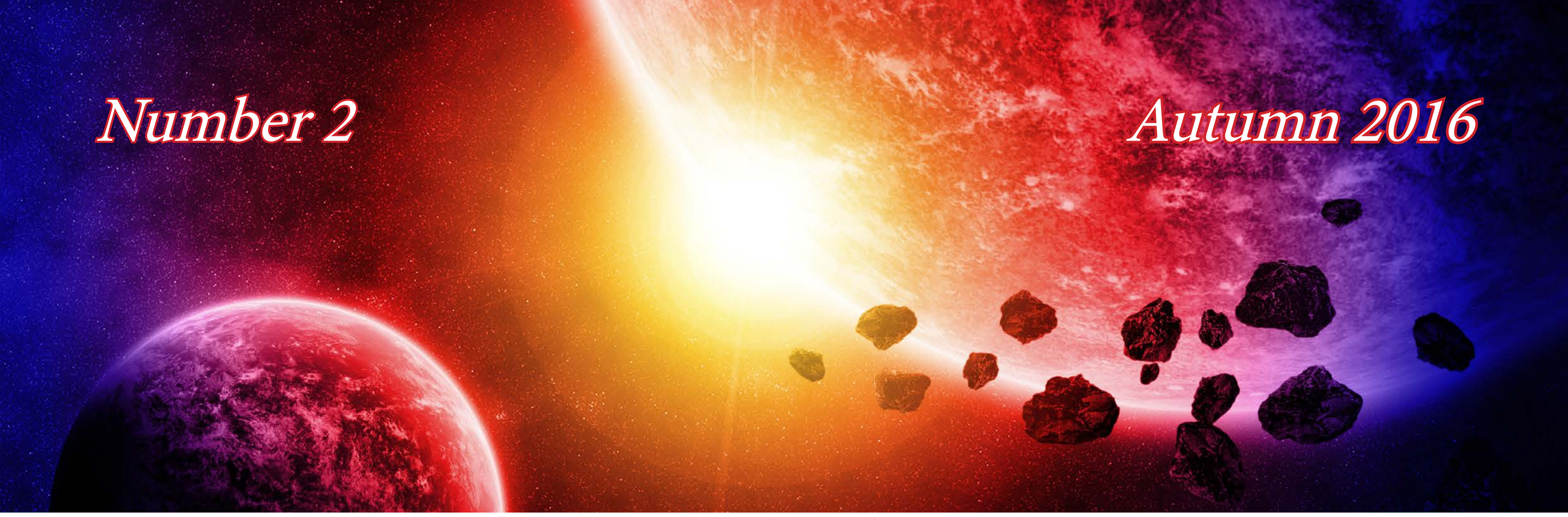

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PROBLEMS FOR JUNIORS

JP.016. Find all triplets (m, n, p) where m, n are two natural numbers and p is a prime number, satisfying the equation:

$$m^4 = 4(p^n - 1).$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.017. Prove the following inequality holds for all positive real numbers a, b, c

$$a^2 + b^2 + c^2 \geq \frac{1}{2}(ab + bc + ca) + \sqrt{\frac{2(a+b+c)(a^3b^3 + b^3c^3 + c^3a^3)}{(a+b)(b+c)(c+a)}}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.018. Let ABC be a triangle with the known normal notations. Prove that for any point P moving on the incircle,

$$5r \leq \frac{PA^2}{h_a} + \frac{PB^2}{h_b} + \frac{PC^2}{h_c} \leq \frac{5}{2}R.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.019. Let k be a positive real number and a, b, c, d be real numbers such that $a + b + c + d = 0$ and $a^2 + b^2 + c^2 + d^2 = 4k$. Prove that $\min(a^{2n} + b^{2n} + c^{2n} + d^{2n}) = 4k^n$, for any real number n with $n > 1$.

Proposed by Cătălin Angelo Ioan - Romania

JP.020. Given $x_1, x_2, \ldots, x_n$ be positive real numbers such that:

$$\sum_{k=1}^{n} x_k = n.$$

If $\alpha, \beta > 0, 4\alpha(n-1)(2\alpha n\sqrt{n} + \beta) > \beta^2\sqrt{n}$. then:

$$\alpha \sum_{i=k}^{n} \frac{1}{x_k} + \frac{\beta}{\sqrt{\sum_{k=1}^{n} x_k^2}} \geq n\alpha + \frac{\beta}{\sqrt{n}}$$

Proposed by Ngo Minh Ngoc Bao - Vietnam

JP.021. Prove that if $x, y, z > 0, xyz = 8$ then:

$$x^3 + y^3 + z^3 \geq 2x\sqrt{y+z} + 2y\sqrt{z+x} + 2z\sqrt{x+y}$$

Proposed by Iuliana Traşcă - Romania

JP.022. Let ABC be an acute triangle with the orthocenter H, inradius r, and circumradius R. Prove that

$$\frac{HA}{\sqrt{bc}} + \frac{HB}{\sqrt{ca}} + \frac{HC}{\sqrt{ab}} \leq \sqrt{2\left(1 + \frac{r}{R}\right)}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.023. Prove that for all positive real numbers a, b, c, d

$$\frac{a}{bc} + \frac{b}{cd} + \frac{c}{da} + \frac{d}{ab} \geq \frac{8}{\sqrt{a^2 + b^2 + c^2 + d^2}}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.024. Given a triangle ABC and let P be any point in its plane. Prove that

$$\frac{PB \cdot PC}{bc} + \frac{PC \cdot PA}{ca} + \frac{PA \cdot PB}{ab} \leq \frac{1}{4}\left(\frac{PA}{h_a} + \frac{PB}{h_b} + \frac{PC}{h_c}\right)^2.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.025. Let $n \geq 2$ be an integer and let a, b, c be positive real numbers such that $ab + bc + ca \leq 1$. Prove that

$$\frac{bc}{(2a^2 + bc)^n} + \frac{ca}{(2b^2 + ca)^n} + \frac{ab}{(2c^2 + ab)^n} \geq 1$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.026. Let a, b, c be non-negative real numbers and let x, y, z be real numbers different from 0, such that $by + cz = x$, $cz + ax = y, ax + by = z$. Prove that

a. $abc \leq \frac{1}{8}$.

b. $\frac{1}{2+a+b} + \frac{1}{2+b+c} + \frac{1}{2+c+a} \leq 1$.

c. $a + b + c \geq 2(ab + bc + ca)$.

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.027. Find all real numbers x satisfying the following equation

$$(x + \{x\})^2 - (x + \{x\}) = 6\lfloor x \rfloor \{x\} - 1$$

where $\lfloor x \rfloor$ and $\{x\}$ denote the integer part and fractional part of x, respectively.

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.028. Prove the following inequality holds for any triangle ABC,

$$\frac{r_b}{r_a}+\frac{r_c}{r_b}+\frac{r_a}{r_c}\geq\sqrt{1+\frac{4R}{r}}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.029. In acute angled $\Delta ABC; L$ - Nagel's point

$$M, M'\in(AB); N, N'\in(AC);$$

$$(M,L,N);(M',L,N')\text{ - collinear points}$$

Prove that:

$$(a+c-b)\Big(\frac{MB}{MA}+\frac{M'B}{M'A}\Big)+(a+b-c)\Big(\frac{NC}{NA}+\frac{N'C}{N'A}\Big)>b+c-a$$

Proposed by Daniel Sitaru - Romania

JP.030. In acute angled $\Delta ABC; \Gamma$ - Gergonne's point

$$\Gamma A'\perp BC; \Gamma B'\perp AC; \Gamma C'\perp BC;$$

$$A'\in(BC); B'\in(AC); C'\in(BC)$$

Prove that:

$$3\sum\Gamma A\geq\sum(\Gamma A'+2r)$$

Proposed by Daniel Sitaru - Romania

PROBLEMS FOR SENIORS

SP.016. Let a,b,c,s,t,u be positive real numbers such that $a+b+c=1$. Prove that:

$$\frac{sa^2+tb^2+uc^2}{sa+tb+uc}+\frac{sb^2+tc^2+ua^2}{sb+tc+ua}+\frac{sc^2+ta^2+ub^2}{sc+ta+ub}\geq 1$$

Proposed by Kunihiko Chikaya - Tokyo - Japan

SP.017. Let $a_k(k=1,2,\ldots,n)$ be a positive real numbers such that

$$\sum_{k=1}^{n}a_k=\frac{n(n+1)}{2}$$

Prove that:

$$\sum_{k=1}^{n}\frac{(k^2-1)a_k+k^2+2k}{a_k^2+a_k+1}\geq\frac{n(n+1)}{2}$$

Proposed by Kunihiko Chikaya - Tokyo - Japan

SP.018. Let a, b and c be real numbers with $a, b, c \geq \frac{1}{2}$ and $a + b + c = 3$. Prove the inequality:

$$\frac{a^2}{b+1} + \frac{b^2}{c+1} + \frac{c^2}{a+1} \geq \frac{a}{b+1} + \frac{b}{c+1} + \frac{c}{a+1}.$$

Proposed by Leonard Giugiuc - Romania

SP.019. Let a, b and c non negative real numbers such that:

$$a + b + c = ab + bc + ac > 0.$$

Prove that

$$\sqrt[n]{a} + \sqrt[n]{b} + \sqrt[n]{c} \geq 2 \cdot \sqrt[n]{2}$$

for any integer n with $n \geq 3$.

Proposed by Leonard Giugiuc - Romania

SP.020. If $x, y, z \in \left(0, \frac{\pi}{2}\right)$, then prove that:

$$\frac{\tan^2 x}{(y+x)^2} + \frac{\tan^2 y}{(z+x)^2} + \frac{\tan^2 z}{(x+y)^2} > \frac{3}{4}$$

Proposed by D. M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

SP.021. If $x, y, z > 0$, then prove that:

$$(x^3y^3 + y^3z^3 + z^3x^3)\left(\frac{1}{(x+y)^5 z} + \frac{1}{(y+z)^5 x} + \frac{1}{(z+x)^5 y}\right) \geq \frac{9}{32}$$

Proposed by D. M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

SP.022. Prove that if $n \in \mathbb{N}; n \geq 2; 0 < a \leq b$ then:

$$\frac{b^{n+1} - a^{n+1}}{n+1} + \frac{ab(b^{n-1} - a^{n-1})}{n-1} \leq (b-a)\sqrt{2(a^{2n} + b^{2n})}$$

Proposed by Daniel Sitaru - Romania

SP.023. Let ABC be a non-obtuse angled triangle. Prove the inequality:

$$\cos A \cos B + \cos A \cos C + \cos B \cos C > 2\sqrt{\cos A \cos B \cos C}.$$

Proposed by Leonard Giugiuc and Daniel Sitaru - Romania

SP.024. Let ABC be a triangle with the centroid G and denote by S_{ABC} its area. Prove that for any point P in the plane:

$$\frac{PA \cdot GA^2}{BC} + \frac{PB \cdot GB^2}{CA} + \frac{PC \cdot GC^2}{AB} \geq \frac{4}{3} S_{ABC}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.025. Let x, y, z be non-negative real numbers such that: $xy+yz+zx+2xyz = 1$. Find the maximum and minimum possible values of:

$$P = xy + yz + zx + x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy}.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.026. Let be $n \in \mathbb{N}^*$. Compute:

$$I = \int_n^{n+1} \frac{\sqrt{n+1-x}}{\sqrt{n+1-x} + \sqrt{x-n}e^{2x-2n-1}} dx$$

Proposed by Daniel Sitaru - Romania

SP.027. Solve the following equation in set of real numbers

$$8^x + 27^{\frac{1}{x}} + 2^{x+1} \cdot 3^{\frac{x+1}{x}} + 2^x \cdot 3^{\frac{2x+1}{x}} = 125$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.028.Compute:

$$L = \lim_{n\to\infty} \frac{1}{n} \int_1^n \frac{x^4 + 4x^3 + 12x^2 + 9x}{(x+3)^5 - x^5 - 243} dx$$

Proposed by Daniel Sitaru - Romania

SP.029. Compute:

$$L = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \int_0^1 \frac{x \sin \pi x}{x + (1-x)k^{1-2x}} dx$$

Proposed by Daniel Sitaru - Romania

SP.030. Prove that:

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos^2 2015x - \cos^2 2016x}{\sin x} dx > 0,0001$$

Proposed by Daniel Sitaru - Romania

UNDERGRADUATE PROBLEMS

UP.016. Compute the limit:

$$\lim_{n\to\infty} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \ln\left(1+\frac{\sin\theta\sec^2\theta}{n}\right)^{\cos\theta}\left(1+\frac{\cos\theta}{n}\right)^{\cot\theta}\left(1+\frac{\cot\theta}{n}\right)^{\sin\theta\sec^2\theta} d\theta.$$

Proposed by Kunihiko Chikaya - Tokyo - Japan

UP.017. Let $\{a_n\}$ be a sequence defined inductively by

$$a_1 = 1, a_{n+1} = \frac{1}{2}a_n + \frac{n^2 - 2n - 1}{n^2(n+1)^2}(n = 1, 2, 3, \ldots).$$

Find the greatest value of n such that $a_1+a_2+\ldots+a_n$ is minimized.

Proposed by Kunihiko Chikaya - Tokyo - Japan

UP.018. Prove that:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\cos x}}{x}dx \leq \frac{\pi}{24}\left(e^{\frac{\pi}{12}} - 1\right)^2 + \frac{\sqrt{2}-1}{4} + \frac{152}{\pi^3}$$

Proposed by Soumitra Mukherjee - Chandar Nagore - India

UP.019. Let be the real number $a \geq b \geq c \geq 1 \geq d \geq e \geq f \geq 0$ such that

$$\sum_{cyc} = 6$$

Then the following statements holds:

$$\text{i)}6 \leq \sum_{cyc} a^2 \leq 18, \quad \text{ii) If} \sum_{cyc} a^2 = q$$

where $q \in [6, 18]$ and q is fixed real number, then:

$$\max\Big(\prod_{cyc} a\Big)$$

its attained for $b = c = 1$ and $d = e = f$.

Proposed by Vasile Cîrtoaje, Leonard Giugiuc - Romania

UP.020. Let a, b, c and d be real numbers such that $\{a, b, c\} \subset (0, 1]$ and $abcd = 1$. Prove that:

$$5(a+b+c+d) + \frac{4}{abc + abd + acd + bcd} \geq 21$$

Proposed by Leonard Giugiuc - Romania

UP.021. Prove that:

$$1 \leq \int_0^1 \frac{dx}{\sqrt{1 - x^2 + x^{2015} - x^{2016}}} \leq \frac{\pi}{2}$$

Proposed by Soumitra Mukherjee - Chandar Nagore - India

UP.022. Let ABC be a triangle with the area S and denote by r, r_a, r_b, r_c inradius, exradii respectively. Prove that:

$$(r^2 + r_a r_b)(r^2 + r_b r_c)(r^2 + r_c r_a) \geq \left(\frac{10}{3}\right)^3 (rS)^2$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

UP.023.

$$\tan^{-1}\frac{1}{\sqrt{2016}} < \int_0^1 \frac{dx}{\sqrt{2016 - x^p - x^q - x^r - x^s - x^t}} \leq \tan^{-1}\frac{1}{\sqrt{2011}}$$

where $p, q, r, s, t \in \mathbb{N}, p < q < r < s < t$ and p is a multiple of 2.

Proposed by Soumitra Mukherjee - Chandar Nagore - India

UP.024. Calculate

$$\sum_{n=2}^{\infty} \operatorname{arctanh}\left(\frac{1}{F_{2n}}\right),$$

where F_n is the n th Fibonacci number.

Proposed by Cornel Ioan Vălean - Romania

UP.025. Compute:

$$\lim_{n\to\infty}\left(\sqrt[3n+3]{(n+1)!} - \sqrt[3n]{n!}\right) \cdot \sqrt[3]{n^2}$$

Proposed by D. M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UP.026. Compute:

$$\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[2n+2]{(2n+1)!!}} - \frac{n}{\sqrt[2n]{(2n-1)!!}}\right)^{\sqrt{n}}$$

Proposed by D. M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UP.027. If $\Gamma : (0, \infty) \to (0, \infty)$ is Euler's function, compute:

$$\lim_{x\to\infty}\left(\frac{x+1}{(\Gamma(x+2))^{\frac{1}{2x+2}}} - \frac{x}{(\Gamma(x+1))^{\frac{1}{2x}}}\right)^{\sqrt{x}}$$

Proposed by D. M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UP.028. Let be $f : (0, \infty) \to (0, \infty)$,
$f(x) = (x+1)^{\frac{(m+1)(x+2)}{x+1}} - x^{\frac{(m+1)(x+1)}{x}}; m \in [0, \infty)$. Compute:

$$\lim_{x\to\infty}\frac{f(x)}{x^m}$$

Proposed by D. M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UP.029. If $x, y \in (0, \infty)$ then:

$$\frac{2}{\pi}\tan^{-1}(x+y)\tan^{-1}\left(\frac{1}{x+y}\right) < \frac{x+y}{4xy+1}$$

Proposed by Daniel Sitaru - Romania

UP.030. If $0 < a < b$ then:

$$b - a + \frac{2}{\pi} \ln \frac{b}{2} \int_a^b \frac{1}{\tan^{-1} x} dx < \frac{\pi}{2} \ln \frac{b}{a} + b - a$$

Proposed by Daniel Sitaru - Romania

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