SOCIETATEA DE ȘTIINȚE MATEMATICE DIN ROMÂNIA
ROMANIAN MATHEMATICAL SOCIETY

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In $\Delta\ ABC$:

$$P: \exists n \in \mathbb{Z}^*;\ \begin{cases} a^2 + (2n+1)a + n^2 = b \\ b^2 + (2n+1)b + n^2 = c \\ c^2 + (2n+1)c + n^2 = a \end{cases}$$

$$Q: r_a + r_b + r_c = w_a + w_b + w_c$$

Prove that: $P \Leftrightarrow Q$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$P: \exists n \in \mathbb{Z}^*;\ \begin{cases} a^2 + (2n+1)a + n^2 = b \rightarrow (1) \\ b^2(2n+1)b + n^2 = c \rightarrow (2) \\ c^2 + (2n+1)c + n^2 = a \rightarrow (3) \end{cases}$$

$$Q: r_a + r_b + r_c = w_a + w_b + w_c \qquad P \Leftrightarrow Q$$

$$(1) + (2) + (3) \Rightarrow \sum a^2 + (2n+1)\sum a + 3n^2 = \sum a$$

$$\Rightarrow \sum a^2 + 2n\sum a + 3n^2 = 0 \Rightarrow 3(n^2) + \left(2\sum a\right)(n) + \sum a^2 = 0$$

$$n \in \mathbb{Z}^* \in \mathbb{R}; \Delta \geq 0 \Rightarrow 4\left(\sum a\right)^2 - 12\sum a^2 \geq 0$$

$$\Rightarrow \left(\sum a\right)^2 - 3\sum a^2 \geq 0 \Rightarrow \sum a^2 + 2\sum ab - 3\sum a^2 \geq 0$$

$$\Rightarrow \sum ab \geq \sum a^2 \quad (1);\ \text{But}, \sum ab \leq \sum a^2 \quad (2)$$

$$(1), (2) \Rightarrow \sum ab = \sum a^2 \Rightarrow \sum (a-b)^2 = 0 \Rightarrow a = b = c$$

$\sum r_a = 4R + r \geq \sum m_a \geq \sum w_a$, equality when $a = b = c$

$P: \Rightarrow a = b = c \Rightarrow \sum r_a = \sum w_a$, that is Q,

meaning $P \Rightarrow Q$

Again, $Q \Rightarrow a = b = c$ and if $a = b = c$,

P definitely holds true $\Rightarrow Q \Rightarrow P$

in conclusion, $P \Leftrightarrow Q$