ROMANIAN MATHEMATICAL MAGAZINE

UP.645 Find:

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{e^{2x}+e^{3y}+e^{4z}}{e^{x+y+z}}dx\,dy\,dz$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{e^{2x}+e^{3y}+e^{4z}}{e^{x+y+z}}dx\,dy\,dz=$$

$$=\int_0^1\int_0^1\int_0^1 e^{2x-x-y-z}\,dx\,dy\,dz+\int_0^1\int_0^1\int_0^1 e^{-x+3y-y-z}\,dx\,dy\,dz+$$

$$+\int_0^1\int_0^1\int_0^1 e^{-x-y+3z}\,dx\,dy\,dz=$$

$$=\left(\int_0^1 e^x dx\right)\left(\int_0^1 e^{-y}dy\right)\left(\int_0^1 e^{-z}dz\right)+\left(\int_0^1 e^{-x}dx\right)\left(\int_0^1 e^{2y}dy\right)\left(\int_0^1 e^{-z}dz\right)+$$

$$+\left(\int_0^1 e^{-x}dx\right)\left(\int_0^1 e^{-y}\,dy\right)\left(\int_0^1 e^{3z}\,dz\right)=$$

$$=(e-1)(e^{-1}-e^0)^2+\left(\frac{1}{2}e^2-\frac{1}{2}\right)(e^{-1}-e^0)^2+\left(\frac{1}{3}e^3-\frac{1}{3}\right)(e^{-1}-e^0)^2=$$

$$=\left(\frac{1}{e}-1\right)^2\left(e-1+\frac{e^2-1}{2}+\frac{e^3-1}{3}\right)=$$

$$=\frac{(1-e)^2}{e^2}\cdot\frac{6e-6+3e^2-3+2e^3-2}{6}=\frac{(1-e)^2(2e^3+3e^2+6e-11)}{6e^2}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

First we split the integrand into three terms

$$\frac{e^{2x}+e^{3y}+e^{4z}}{e^{x+y+z}}=e^xe^{-y}e^{-z}+e^{-x}e^{2y}e^{-z}+e^{-x}e^{-y}e^{3z}.$$

Each term is a product of functions of a single variable and the domain of integration is the cube $[0,1]^3$. Therefore, by Fubini's theorem, the integral of each term is the product of three one-variable integrals. We need the following integrals:

$$\int_0^1 e^{-t}\,dt=\frac{e-1}{e},\quad\int_0^1 e^t\,dt=e-1,$$

$$\int_0^1 e^{2t}\,dt = \frac{e^2-1}{2},\quad \int_0^1 e^{3t}\,dt = \frac{e^3-1}{3}.$$

Now, we evaluate the first term:

$$\int_0^1\int_0^1\int_0^1 e^x e^{-y} e^{-z}\,dx\,dy\,dz = (e-1)\left(\frac{e-1}{2}\right)^2 = \frac{(e-1)^3}{e^2}.$$

The second term:

$$\int_0^1\int_0^1\int_0^1 e^{-x}\,e^{2y}\,e^{-z}\,dx\,dy\,dz = \frac{e^2-1}{2}\left(\frac{e-1}{e}\right)^2 = \frac{(e^2-1)(e-1)^2}{2e^2}.$$

The third term:

$$\int_0^1\int_0^1\int_0^1 e^{-x}e^{-y}e^{3z}\,dx\,dy\,dz = \frac{e^3-1}{3}\left(\frac{e-1}{e}\right)^2 = \frac{(e^3-1)(e-1)^2}{3e^2}.$$

Adding up, yields

$$\Omega = \frac{(e-1)^2}{e^2}\left[(e-1)+\frac{e^2-1}{2}+\frac{e^3-1}{3}\right] =$$

$$= \frac{(e-1)^2}{6e^2}[6(e-1)+3(e^2-1)+2(e^3-1)].$$

Hence,

$$\Omega = \frac{(e-1)^2(2e^3+3e^2+6e-11)}{6e^2}.$$