ROMANIAN MATHEMATICAL MAGAZINE

UP.644 Let be:

$$D=\{(x,y,z)|x,y,z\in\mathbb{R};0\le x\le 1;0\le y\le x;0\le z\le xy\}$$

Find:

$$\Omega=\iiint_D xyz\,dx\,dy\,dz$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

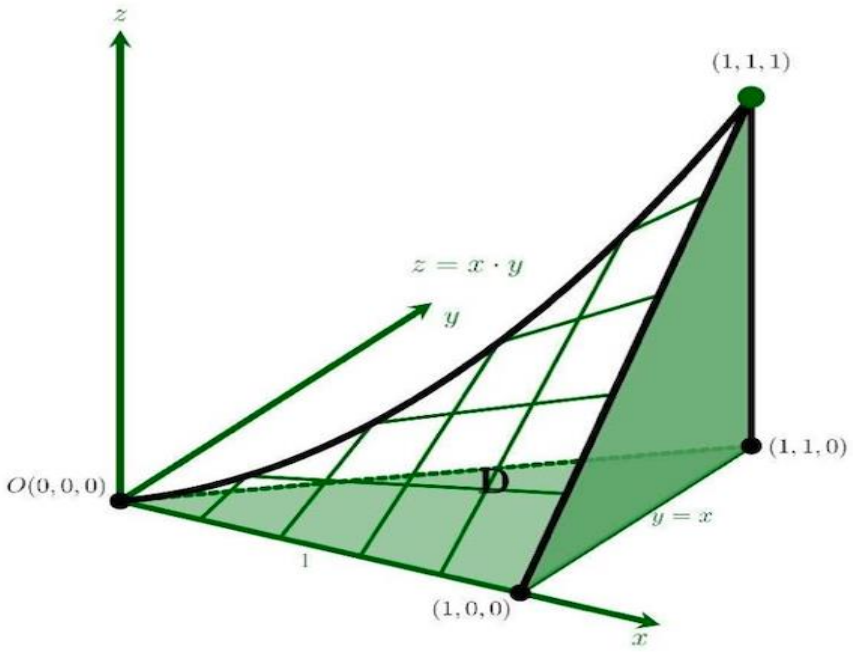
$$\Omega=\iiint_D xyz\,dx\,dy\,dz=\int_0^1\left(\int_0^x\left(\int_0^{xy}xyz\,dz\right)dy\right)dx=$$

$$=\int_0^1\left(\int_0^x\left(xy\cdot\frac{z^2}{2}\Big|_0^{xy}\right)dy\right)dz=\int_0^1\left(\int_0^x\frac{(xy)^3}{x}dy\right)dx=\int_0^1\left(\frac{x^3}{2}\int_0^x y^3\,dy\right)dx=$$

$$=\int_0^1\left(\frac{x^3}{2}\cdot\frac{y^4}{4}\Big|_0^x\right)dx=\int_0^1\left(\frac{x^3}{2}\cdot\frac{x^4}{4}\right)dx=\frac{1}{8}\int_0^1 x^7\,dx=\frac{1}{8}\cdot\frac{x^8}{8}\Big|_0^1=\frac{1}{64}$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$D=\{(x,y,z)\in\mathbb{R}^3:0\le x\le 1,0\le y\le x,0\le z\le x\cdot y\}$$



$$\Omega=\int\int\int_D x\cdot y\cdot z\,dx\,dy\,dz,\qquad \Omega=\int_0^1\left(\int_0^x\left(\int_0^{x\cdot y}x\cdot y\cdot z\,dz\right)dy\right)dx$$

$$\Omega=\int_0^1 x\cdot\left(\int_0^x y\cdot\left(\int_0^{x\cdot y}z\,dz\right)dy\right)dx$$

$$\int_0^{x\cdot y}z\,dz=\left[\frac{z^{1+1}}{1+1}\right]_0^{x\cdot y}=\left[\frac{z^2}{2}\right]_0^{x\cdot y}=\frac{(x\cdot y)^2}{2}-\frac{0^2}{2}=\frac{x^2\cdot y^2}{2}-0=\frac{x^2\cdot y^2}{2}$$

$$\Omega=\int_0^1 x\cdot\left(\int_0^x y\cdot\left(\frac{x^2\cdot y^2}{2}\right)dy\right)dx$$

$$\Omega=\int_0^1 x\cdot\left(\int_0^x\frac{x^2\cdot y^{1+2}}{2}dy\right)dx=\int_0^1 x\cdot\left(\int_0^x\frac{x^2\cdot y^3}{2}dy\right)dx$$

$$\Omega=\int_0^1 x\cdot\left(\frac{x^2}{2}\cdot\int_0^x y^3\,dy\right)dx$$

$$\int_0^x y^3\,dy=\left[\frac{y^{3+1}}{3+1}\right]_0^x=\left[\frac{y^4}{4}\right]_0^x=\frac{x^4}{4}-\frac{0^4}{4}=\frac{x^4}{4}-0=\frac{x^4}{4},\qquad \Omega=\int_0^1 x\cdot\left(\frac{x^2}{2}\cdot\frac{x^4}{4}\right)dx$$

$$\Omega=\int_0^1 x\cdot\left(\frac{x^2\cdot x^4}{2\cdot 4}\right)dx=\int_0^1 x\cdot\left(\frac{x^{2+4}}{8}\right)dx=\int_0^1 x\cdot\frac{x^6}{8}dx$$

$$\Omega=\int_0^1\frac{x^{1+6}}{8}dx=\int_0^1\frac{x^7}{8}dx=\frac{1}{8}\int_0^1 x^7\,dx$$

$$\int_0^1 x^7\,dx=\left[\frac{x^{7+1}}{7+1}\right]_0^1=\left[\frac{x^8}{8}\right]_0^1=\frac{1^8}{8}-\frac{0^8}{8}=\frac{1}{8}-0=\frac{1}{8}$$

$$\Omega=\frac{1}{8}\cdot\frac{1}{8}=\frac{1\cdot 1}{8\cdot 8}=\frac{1}{64}\to\Omega=\frac{1}{64}$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\begin{cases}x=x\\ y=u\cdot x\\ z=v\cdot x\cdot y=v\cdot u\cdot x^2\end{cases}$$

$$\begin{cases}0\le x\le 1\\ 0\le y\le x\Rightarrow 0\le u\cdot x\le x\Rightarrow 0\le u\le 1\\ 0\le z\le x\cdot y\Rightarrow 0\le v\cdot u\cdot x^2\le u\cdot x^2\Rightarrow 0\le v\le 1\end{cases}$$

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$$J=\frac{\partial(x,y,z)}{\partial(x,u,v)}=\begin{vmatrix}\frac{\partial x}{\partial x} & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}\end{vmatrix}$$

$$\begin{cases}\frac{\partial x}{\partial x}=1, & \frac{\partial x}{\partial u}=0, & \frac{\partial x}{\partial v}=0\\ \frac{\partial y}{\partial x}=u, & \frac{\partial y}{\partial u}=x, & \frac{\partial y}{\partial v}=0\\ \frac{\partial z}{\partial x}=2\cdot u\cdot v\cdot x, & \frac{\partial z}{\partial u}=v\cdot x^2, & \frac{\partial z}{\partial v}=u\cdot x^2\end{cases}$$

$$J=\begin{vmatrix}1 & 0 & 0\\ u & x & 0\\ 2uvx & vx^2 & ux^2\end{vmatrix}$$

$$J=1\cdot\begin{vmatrix}x & 0\\ vx^2 & ux^2\end{vmatrix}-0\cdot\begin{vmatrix}u & 0\\ 2uvx & ux^2\end{vmatrix}+0\cdot\begin{vmatrix}u & x\\ 2uvx & vx^2\end{vmatrix}$$

$$J=1\cdot(x\cdot ux^2-0\cdot vx^2)-0+0=x\cdot ux^2=u\cdot x^3$$

$$x\cdot y\cdot z=x\cdot(u\cdot x)\cdot(v\cdot u\cdot x^2)=u^2\cdot v\cdot x^4$$

$$\Omega=\int_0^1\int_0^1\int_0^1(u^2\cdot v\cdot x^4)\cdot|J|dx\,du\,dv$$

$$\Omega=\int_0^1\int_0^1\int_0^1(u^2\cdot v\cdot x^4)\cdot(u\cdot x^3)dx\,du\,dv,\qquad \Omega=\int_0^1\int_0^1\int_0^1 x^7\cdot u^3\cdot v\,dx\,du\,dv$$

$$\Omega=\left(\int_0^1 x^7dx\right)\cdot\left(\int_0^1 u^3\,du\right)\cdot\left(\int_0^1 v\,dv\right)$$

$$\int_0^1 x^7\,dx=\left[\frac{x^{7+1}}{7+1}\right]_0^1=\left[\frac{x^8}{8}\right]_0^1=\frac{1^8}{8}-\frac{0^8}{8}=\frac{1}{8}-0=\frac{1}{8}$$

$$\int_0^1 u^3\,du=\left[\frac{u^{3+1}}{3+1}\right]_0^1=\left[\frac{u^4}{4}\right]_0^1=\frac{1^4}{4}-\frac{0^4}{4}=\frac{1}{4}-0=\frac{1}{4}$$

$$\int_0^1 v\,dv=\left[\frac{v^{1+1}}{1+1}\right]_0^1=\left[\frac{v^2}{2}\right]_0^1=\frac{1^2}{2}-\frac{0^2}{2}=\frac{1}{2}-0=\frac{1}{2}$$

$$\Omega=\frac{1}{8}\cdot\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8\cdot4\cdot2}\to\Omega=\frac{1}{64}$$

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Solution 4 by Jose Luis Diaz Barrero-Spain

The region D is described by

$$0 \le x \le 1, \;\; 0 \le y \le x, \;\; 0 \le z \le xy,$$

So, by Fubini's theorem, the triple integral can be written as the iterated integral

$$\Omega = \int_0^1 \left(\int_0^x \left(\int_0^{xy} xyz \, dz \right) dy \right) dx.$$

Since x and y are constant with respect to z, then

$$\int_0^{xy} xyz \, dz = xy \left[\frac{z^2}{2} \right]_{z=0}^{z=xy} = xy \cdot \frac{(xy)^2}{2} = \frac{x^2 y^3}{2}.$$

Now x is constant with respect to y, hence

$$\int_0^x \frac{x^3 y^3}{2} dy = \frac{x^3}{2} \left[\frac{y^4}{4} \right]_{y=0}^{y=x} = \frac{x^3}{2} \cdot \frac{x^4}{4} = \frac{x^7}{8}.$$

Finally,

$$\Omega = \int_0^1 \frac{x^7}{8} dx = \frac{1}{8} \left[\frac{x^8}{8} \right]_{x=0}^{x=1} = \frac{1}{8} \cdot \frac{1}{8} = \frac{1}{64}.$$

Therefore,

$$\Omega = \iiint_D xyz \, dx \, dy \, dz = \frac{1}{64}.$$