ROMANIAN MATHEMATICAL MAGAZINE

UP.643 Find:

$$\Omega = \int_0^\infty \int_0^\infty \int_0^\infty (2x + 3y + 5z)e^{-x^2-y^2-z^2}\, dx\, dy\, dz$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\Omega = 2\int_0^\infty \int_0^\infty \int_0^\infty xe^{-x^2-y^2-z^2} dx\, dy\, dz + 3\int_0^\infty \int_0^\infty \int_0^\infty ye^{-x^2-y^2-z^2} dx\, dy\, dz +$$

$$+5\int_0^\infty \int_0^\infty \int_0^\infty ze^{-x^2-y^2-z^2} dx\, dy\, dz = 10\int_0^\infty \int_0^\infty \int_0^\infty xe^{-x^2-y^2-z^2} dx\, dy\, dz =$$

$$= 10\left(\int_0^\infty xe^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right)\left(\int_0^\infty e^{-z^2} dz\right) = \frac{10}{-2}\int_0^\infty (e^{-x^2})' dx \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\sqrt{\pi}}{2} =$$

$$= \frac{-5\pi}{4} e^{-x^2}\Big|_0^\infty = -\frac{5\pi}{4}(e^{-\infty} - e^0) = -\frac{5\pi}{4}(0 - 1) = \frac{5\pi}{4}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

Let

$$G = \int_0^\infty e^{-t^2}\, dt, M = \int_0^\infty te^{-t^2}\, dt.$$

Using polar coordinates in the first quadrant,

$$G^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\, dr\, d\theta = \frac{\pi}{2} \cdot \frac{1}{2} = \frac{\pi}{4},$$

so $G = \frac{\sqrt{\pi}}{2}$**, as it is well-known. Moreover:** $M = \left[-\frac{1}{2}e^{-t^2}\right]_0^\infty = \frac{1}{2}.$

Since $e^{-x^2-y^2-z^2} = e^{-x^2}e^{-y^2}e^{-z^2}$ **and the inegrand is nonnegative, by Tonelli's theorem we may integrate term by term and factor each term. For instance,**

$$\int_0^\infty \int_0^\infty \int_0^\infty xe^{-x^2}e^{-y^2}e^{-z^2}\, dx\, dy\, dz = MG^2 = \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{8},$$

and, by symmetry, the terms in y **and** z **give the same value. Finally, we have**

$$\Omega = (2 + 3 + 5)MG^2 = 10 \cdot \frac{\pi}{8} = \frac{5\pi}{4}.$$