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UP.642 Find the line integral of the total differentials:

$$\Omega = \int_{(0,0,0)}^{\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right)} 2\cos x\,dx + 3\cos y\;dy + 5\cos z\;dz$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$2\cos x\;dx + 3\cos y\;dy + 5\cos z\;dz = dU$$

$$\frac{\partial U}{\partial x} = 2\cos x; \frac{\partial U}{\partial y} = 3\cos y; \frac{\partial U}{\partial z} = 5\cos z$$

$$U(x,y,z) = 2\sin x + 3\sin y + 5\sin z$$

$$U\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right) = 2\cdot 1 + 3\cdot 1 + 5\cdot 1 = 10$$

$$U(0,0,0) = 0$$

$$\Omega = U\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right) - U(0,0,0) = 10 - 0 = 10$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\Omega = \int_{(0,0,0)}^{\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right)} 2\cos x\,dx + 3\cos y\,dy + 5\cos z\,dz, \qquad C = C_1 \cup C_2 \cup C_3$$

$$P_0(0,0,0) \to P_1\left(\frac{\pi}{2},0,0\right) \to P_2\left(\frac{\pi}{2},\frac{\pi}{2},0\right) \to P_3\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$C_1: P_0(0,0,0) \to P_1\left(\frac{\pi}{2},0,0\right), \qquad \begin{cases} x \in \left[0,\frac{\pi}{2}\right] \\ y = 0 \Rightarrow dy = 0 \\ z = 0 \Rightarrow dz = 0 \end{cases}$$

$$\Omega_1 = \int_0^{\frac{\pi}{2}} 2\cos x\,dx + 3\cos(0) + 5\cos(0)\cdot 0 = \int_0^{\frac{\pi}{2}} 2\cos x\,dx$$

$$= [2\sin x]_0^{\frac{\pi}{2}} = 2\sin\left(\frac{\pi}{2}\right) - 2\sin(0) = 2(1) - 2(0) = 2$$

$$C_2: P_1\left(\frac{\pi}{2},0,0\right) \to P_2\left(\frac{\pi}{2},\frac{\pi}{2},0\right), \qquad \begin{cases} x = \frac{\pi}{2} \Rightarrow dx = 0 \\ y \in \left[0,\frac{\pi}{2}\right] \\ z = 0 \Rightarrow dz = 0 \end{cases}$$

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$$\Omega_2 = \int_0^{\frac{\pi}{2}} 2\cos\left(\frac{\pi}{2}\right)\cdot 0 + 3\cos y\, dy + 5\cos(0)\cdot 0 = \int_0^{\frac{\pi}{2}} 3\cos y\, dy$$

$$= [3\sin y]_0^{\frac{\pi}{2}} = 3\sin\left(\frac{\pi}{2}\right) - 3\sin(0) = 3(1) - 3(0) = 3$$

$$C_3: P_2\left(\frac{\pi}{2},\frac{\pi}{2},0\right) \to P_3\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right), \qquad \begin{cases} x = \frac{\pi}{2} \Rightarrow dx = 0 \\ y = \frac{\pi}{2} \Rightarrow dy = 0 \\ z \in \left[0,\frac{\pi}{2}\right] \end{cases}$$

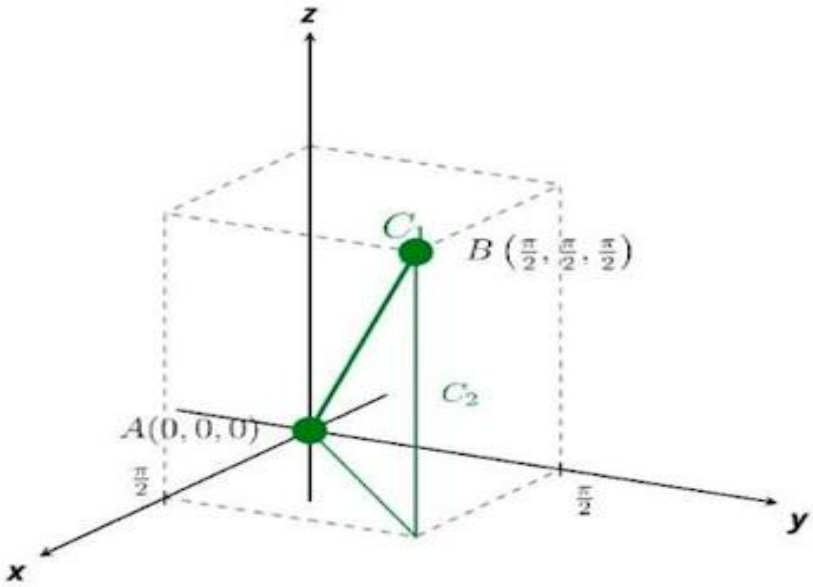
$$\Omega_3 = \int_0^{\frac{\pi}{2}} 2\cos\left(\frac{\pi}{2}\right)\cdot 0 + 3\cos\left(\frac{\pi}{2}\right)\cdot 0 + 5\cos z\, dz = \int_0^{\frac{\pi}{2}} 5\cos z\, dz$$

$$= [5\sin z]_0^{\frac{\pi}{2}} = 5\sin\left(\frac{\pi}{2}\right) - 5\sin(0) = 5(1) - 5(0) = 5$$

$$\Omega = \Omega_1 + \Omega_2 + \Omega_3 = 2 + 3 = 5, \qquad \Omega = 10$$

Observation:

$$\nabla \times \vec{F} = \vec{0} \Rightarrow \int_{C_1} \omega = \int_{C_2} \omega = f(B) - f(A)$$



$$\Omega = \int_C df = f(B) - f(A) = 10$$

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Solution 3 by Jose Luis Diaz Barrero-Spain

Write the integrand as $P\,dx + Q\,dy + R\,dz$, where

$$P = 2\cos x, \quad Q = 3\cos y, \quad R = 5\cos z.$$

Each coefficient depends only on its own variable, so

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 0, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = 0, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = 0.$$

Thus curl $(P, Q, R) = 0$ and the form is closed. Since $\mathbb{R}^3$ is star-shaped, Poincare's lemma ensures that the form is exact, that is $P\,dx + Q\,dy + R\,dz = dF$ for some F of class C^1 on $\mathbb{R}^3$.

Now, we look for F with $F_x = P, F_y = Q, F_z = R$. Integrating $F_x = 2\cos x$ with respect to x,

$$F(x, y, z) = 2\sin x + g(y, z).$$

From $F_y = g_y = 3\cos y$ we get $g(y, z) = 3\sin y + h(z)$, and from $F_z = h'(z) = 5\cos z$ we get $h(z) = 5\sin z + C$. Hence

$$F(x, y, z) = 2\sin x + 3\sin y + 5\sin z + C$$

Let $\gamma(t), a \le t \le b$, be any piecewise C^1 curve joining $A = (0, 0, 0)$ to $B = \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$. By the chain rule,

$$\int_\gamma dF = \int_a^b \nabla F\big(\gamma(t)\big) \cdot \gamma'(t) dt = \int_a^b \frac{d}{dt} F\big(\gamma(t)\big) dt = F(B) - F(A),$$

so the integral depends only on the endpoints.

Finally, we have

$$\Omega = F\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) - F(0, 0, 0) = (2 + 3 + 5) - 0 = 10.$$

Along the segment $\gamma(t) = (t, t, t), 0 \le t \le \frac{\pi}{2}$, we have $dx = dy = dz = dt$, and

$$\Omega = \int_0^{\frac{\pi}{2}} (2 + 3 + 5)\cos t\, dt = 10[\sin t]_0^{\frac{\pi}{2}} = 10.$$