ROMANIAN MATHEMATICAL MAGAZINE

UP.641 Find the line integral of the total differentials:

$$\Omega = \int_{(3,4,5)}^{(6,8,10)} \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\frac{x}{\sqrt{x^2+y^2+z^2}}dx + \frac{y}{\sqrt{x^2+y^2+z^2}}dy + \frac{z}{\sqrt{x^2+y^2+z^2}}dz = dU$$

$$\left.\begin{aligned} \frac{\partial U}{\partial x} &= \frac{x}{\sqrt{x^2+y^2+z^2}} \\ \frac{\partial U}{\partial y} &= \frac{y}{\sqrt{x^2+y^2+z^2}} \\ \frac{\partial U}{\partial z} &= \frac{z}{\sqrt{x^2+y^2+z^2}} \end{aligned}\right\} \Rightarrow U(x,y,z) = \sqrt{x^2+y^2+z^2}$$

$$\Omega = U(6,8,10) - U(3,4,5) =$$

$$= \sqrt{6^2+8^2+10^2} - \sqrt{3^2+4^2+5^2} = \sqrt{200} - \sqrt{50} = 10\sqrt{2} - 5\sqrt{2} = 5\sqrt{2}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

Find the integral of the total differential

$$\Omega = \int_{(3,4,5)}^{(6,8,10)} \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}}.$$

Let $r = \sqrt{x^2+y^2+z^2}$**. For** $(x,y,z) \neq (0,0,0)$ **we have**

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \frac{\partial r}{\partial y} = \frac{y}{r}, \qquad \frac{\partial r}{\partial z} = \frac{z}{r},$$

hence

$$\frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}} = dr,$$

that is, the integrand is the total differential of the potential $F = r$**. Since** r **is of class** C^1 **on** $D = \mathbb{R}^3 \setminus \{(0,0,0)\}$ **(it is not differentiable at the origin), the form is exact on** D**. Consequently, if** $\gamma(t), a \leq t \leq b$ **is any piecewise** C^1 **curve in** D **joining** A **to** B**, the chain rule gives**

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$$\int_{\gamma} dr = \int_a^b \nabla r\big(\gamma(t)\big) \cdot \gamma'(t) dt = \int_a^b \frac{d}{dt} r\big(\gamma(t)\big)\, dt = r(B) - r(A),$$

so the integral depends only on the endpoints, provided the path avoids the origin. For instance, the segment $\gamma(t) = t(3,4,5), 1 \le t \le 2$, lies in D, and along it $r = 5\sqrt{2}t$, which gives

$$\Omega = \int_1^2 5\sqrt{2}\, dt = r(2) - r(1) = 5\sqrt{2}.$$