ROMANIAN MATHEMATICAL MAGAZINE

UP.640 Compute the following sum

$$\sum_{n=1}^{\infty} \frac{x^n}{1 - x^n - x^{n+1} + x^{2n+1}}$$

Proposed by José Luis Díaz – Barrero – Spain

Solution by proposer

Notice that

$$1 - x^n - x^{n+1} + x^{2n+1} = (1 - x^n) - x^{n+1}(1 - x^n) = (1 - x^n)(1 - x^{n+1}).$$

So each term of the series becomes

$$\frac{x^n}{(1 - x^n)(1 - x^{n+1})}.$$

Consider the difference

$$\frac{1}{1 - x^n} - \frac{1}{1 - x^{n+1}} = \frac{(1 - x^{n+1}) - (1 - x^n)}{(1 - x^n)(1 - x^{n+1})} = \frac{x^n - x^{n+1}}{(1 - x^n)(1 - x^{n+1})} = \frac{x^n(1 - x)}{(1 - x^n)(1 - x^{n+1})}.$$

Dividing both sides by $(1 - x)$ (assuming $x \neq 1$), we have

$$\frac{x^n}{(1 - x^n)(1 - x^{n+1})} = \frac{1}{1 - x}\left(\frac{1}{1 - x^n} - \frac{1}{1 - x^{n+1}}\right).$$

So the general term is exactly a telescoping difference.

Next, we sum the telescoping series:

$$S_N = \sum_{n=1}^{N} \frac{x^n}{(1 - x^n)(1 - x^{n+1})} = \frac{1}{1 - x}\sum_{n=1}^{N}\left(\frac{1}{1 - x^n} - \frac{1}{1 - x^{n+1}}\right)$$

$$= \frac{1}{1 - x}\left(\frac{1}{1 - x} - \frac{1}{1 - x^{N+1}}\right).$$

For $|x| < 1$, we have $x^{N+1} \to 0$, so $\frac{1}{1 - x^{N+1}} \to 1$. Therefore

$$S_\infty = \frac{1}{1 - x}\left(\frac{1}{1 - x} - 1\right) = \frac{1}{1 - x} \cdot \frac{x}{1 - x} = \frac{x}{(1 - x)^2}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{x^n}{1 - x^n - x^{n+1} + x^{2n+1}} = \frac{x}{(1 - x)^2}, \quad |x| < 1.$$