ROMANIAN MATHEMATICAL MAGAZINE

UP.639 Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k}\left(i^2+\frac{i}{3}-\frac{1}{12}\right)\right]^{-1}$$

Proposed by Omkumar Rao-India

Solution 1 by proposer

$$\sum_{i=1}^{k}\left(i^2+\frac{i}{3}-\frac{1}{12}\right)=\frac{k(k+1)(2k+1)}{6}+\frac{k(k+1)}{6}-\frac{k}{12}=\frac{k(2k+1)(2k+3)}{12}$$

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{12}{k(2k+1)(2k+3)}=$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{4}{k}-\frac{12}{2k+1}+\frac{4}{2k+3}\right)=4\lim_{n\to\infty}H_n-12\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2k+1}+4\underbrace{\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2k+3}}_{\sum_{k=2}^{n+1}\frac{1}{2k+1}}$$

$$\Omega=4\lim_{n\to\infty}H_n-12\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2k+1}+4\lim_{n\to\infty}\sum_{k=2}^{n+1}\frac{1}{2k+1}$$

$$=4\lim_{n\to\infty}H_n-\frac{12}{3}+\frac{4}{2n+3}-8\lim_{n\to\infty}\sum_{k=2}^{n}\frac{1}{2k+1}$$

$$\Omega=4\lim_{n\to\infty}H_n-\frac{12}{3}+\lim_{n\to\infty}\frac{4}{2n+3}-8\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{2k+1}-\frac{1}{3}\right)$$

$$=4\lim_{n\to\infty}H_n-\frac{4}{3}+\underbrace{\lim_{n\to\infty}\frac{4}{2n+3}}_{0}-8\lim_{n\to\infty}\underbrace{\sum_{k=1}^{n}\frac{1}{2k+1}}_{H_{2n+1}-1-\frac{1}{2}H_n}$$

$$\Omega=4\lim_{n\to\infty}H_n-\frac{4}{3}-8\lim_{n\to\infty}\left(H_{2n+1}-1-\frac{1}{2}H_n\right)=8\lim_{n\to\infty}H_n-8\lim_{n\to\infty}H_{2n+1}+\frac{20}{3}$$

$$\Omega\overset{H_n=\ln(n)+\gamma+O\left(\frac{1}{n}\right)}{=}8\lim_{n\to\infty}\left(\left(\ln(n)+\gamma+O\left(\frac{1}{n}\right)\right)-\left(\ln(2n+1)+\gamma+O\left(\frac{1}{2n+1}\right)\right)\right)+\frac{20}{3}$$

$$\Omega=8\lim_{n\to\infty}\ln\left(\frac{n}{2n+1}\right)+\frac{20}{3}=\frac{20}{3}-8\ln 2$$

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k}\left(i^2+\frac{i}{3}-\frac{1}{12}\right)\right]^{-1}=\frac{20}{3}-8\ln 2$$

Solution 2 by Jose Luis Diaz Barrero-Spain

First we compute the inner sum:

$$\sum_{i=1}^{k}\left(i^2+\frac{i}{3}-\frac{1}{12}\right)=\frac{k(k+1)(2k+1)}{6}+\frac{k(k+1)}{6}-\frac{k}{12}$$

$$=\frac{k(4k^2+8k+3)}{12}=\frac{k(2k+1)(2k+3)}{12}.$$

Decomposing its inverse into partial fractions, we get

$$\frac{12}{k(2k+1)(2k+3)}=\frac{4}{k}-\frac{12}{2k+1}+\frac{4}{2k+3}=8\left(\frac{1}{2k}-\frac{1}{2k+1}\right)-4\left(\frac{1}{2k+1}-\frac{1}{2k+3}\right)$$

Since

$$\sum_{k=1}^{\infty}\left(\frac{1}{2k}-\frac{1}{2k+1}\right)=\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots=1-\ln 2$$

and

$$\sum_{k=1}^{\infty}\left(\frac{1}{2k+1}-\frac{1}{2k+3}\right)=\frac{1}{3}$$

(telescoping), it follows that

$$\Omega=8(1-\ln 2)-\frac{4}{3}=\frac{20}{3}-8\ln 2.$$