ROMANIAN MATHEMATICAL MAGAZINE

UP.638 Compute the square root of the following matrix

$$X=\frac{1}{14}\begin{pmatrix}41 & 23 & -25 & 1\\ 19 & 46 & -29 & 2\\ 21 & -7 & 7 & 7\\ 17 & -6 & -23 & 46\end{pmatrix}$$

Proposed by José Luis Díaz – Barrero – Spain

Solution 1 by proposer

We are given

$$X=\frac{1}{14}\begin{pmatrix}41 & 23 & -25 & 1\\ 19 & 46 & -29 & 2\\ 21 & -7 & 7 & 7\\ 17 & -6 & -23 & 46\end{pmatrix}=\begin{pmatrix}\frac{41}{14} & \frac{23}{14} & -\frac{25}{14} & \frac{1}{14}\\ \frac{19}{14} & \frac{23}{7} & -\frac{29}{14} & \frac{1}{7}\\ \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ \frac{17}{14} & -\frac{3}{7} & -\frac{23}{14} & \frac{23}{7}\end{pmatrix}$$

and we want a matrix Y such that $Y^2=X$. We proceed using the standard diagonalization approach. We carry out the following steps:

1. Computing the characteristic polynomial and eigenvalues:

Computing $\det(X-\lambda I)$ (via cofactor expansion or row reduction of the symbolic matrix $X-\lambda I$) yields factored characteristic polynomial

$$\det(X-\lambda I)=(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4).$$

Hence the eigenvalues of X are

$$\lambda_1=1,\lambda_2=2,\lambda_3=3,\lambda_4=4.$$

All four eigenvalues are real, distinct, and strictly positive. Since X is a 4×4 matrix with 4 distinct eigenvalues, it is automatically diagonalizable. Because the spectrum is positive, a real square root of X exists.

2. Computing the eigenvectors: For each eigenvalue λ, we solve the homogeneous system $(x-\lambda I)v=0$, i.e. find $\ker(X-\lambda I)$. In every case the coefficient matrix has rank 3, confirming a one-dimensional eigenspace, consistent with the eigenvalues being simple.

$$\lambda=1:$$

ROMANIAN MATHEMATICAL MAGAZINE

$$X-I=\begin{pmatrix}\frac{27}{14} & \frac{23}{14} & -\frac{25}{14} & \frac{1}{14}\\ \frac{19}{14} & \frac{16}{7} & -\frac{29}{14} & \frac{1}{7}\\ \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\ \frac{17}{14} & -\frac{3}{7} & -\frac{23}{14} & \frac{16}{7}\end{pmatrix},\ker(X-I)=span\{(1,2,3,2)^T\}.$$

$\lambda=2$:

$$X-2I=\begin{pmatrix}\frac{13}{14} & \frac{23}{14} & -\frac{25}{14} & \frac{1}{14}\\ \frac{19}{14} & \frac{9}{7} & -\frac{29}{14} & \frac{1}{7}\\ \frac{3}{2} & -\frac{1}{2} & -\frac{3}{2} & \frac{1}{2}\\ \frac{17}{14} & -\frac{3}{7} & -\frac{23}{14} & \frac{9}{7}\end{pmatrix},\ker(X-2I)=span\{(2,1,2,1)^T\}.$$

$\lambda=3$:

$$X-3I=\begin{pmatrix}-\frac{1}{14} & \frac{23}{14} & -\frac{25}{14} & \frac{1}{14}\\ \frac{19}{14} & \frac{2}{7} & \frac{29}{14} & \frac{1}{7}\\ \frac{3}{2} & -\frac{1}{2} & -\frac{5}{2} & \frac{1}{2}\\ \frac{17}{14} & -\frac{3}{7} & -\frac{23}{14} & \frac{2}{7}\end{pmatrix},\ker(X-3I)=span\{(1,1,1,3)^T\}.$$

$\lambda=4$:

$$X-4I=\begin{pmatrix}-\frac{15}{14} & \frac{23}{14} & -\frac{25}{14} & \frac{1}{14}\\ \frac{19}{14} & -\frac{5}{7} & -\frac{29}{14} & \frac{1}{7}\\ \frac{3}{2} & -\frac{1}{2} & -\frac{7}{2} & \frac{1}{2}\\ \frac{17}{14} & -\frac{3}{7} & -\frac{23}{14} & -\frac{5}{7}\end{pmatrix},\ker(X-4I)=span\{(3,3,1,1)^T\}.$$

3. Assemble the eigenvector matrix P and invert it: Placing the eigenvectors as columns, in the order $\lambda=1,2,3,4$, we have

$$P=\begin{pmatrix}1 & 2 & 1 & 3\\ 2 & 1 & 1 & 3\\ 3 & 2 & 1 & 1\\ 2 & 1 & 3 & 1\end{pmatrix}.$$

One computes $\det(P)=-28\neq 0$, confirming P is invertible (as it must be, since the eigenvectors of a matrix with distinct eigenvalues are automatically linearly independent). Its inverse is

$$P^{-1}=\frac{1}{14}\begin{pmatrix}-6 & 5 & 4 & -1\\ 8 & -9 & 4 & -1\\ 1 & -2 & -3 & 6\\ 1 & 5 & -3 & -1\end{pmatrix}=\begin{pmatrix}-\frac{3}{7} & \frac{5}{14} & \frac{2}{7} & -\frac{1}{14}\\ \frac{4}{7} & -\frac{9}{14} & \frac{2}{7} & -\frac{1}{14}\\ \frac{1}{14} & -\frac{1}{7} & -\frac{3}{14} & \frac{3}{7}\\ \frac{1}{14} & \frac{5}{14} & -\frac{3}{14} & -\frac{1}{14}\end{pmatrix}.$$

By construction,

$$X=P\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 4\end{pmatrix}P^{-1}.$$

(This can be verified directly by multiplying out the right-hand side and checking it matches the given X.)

4. Take the square root of the diagonal part: Since every eigenvalue is positive, we may take positive square roots:

$$D^{\frac{1}{2}}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \sqrt{2} & 0 & 0\\ 0 & 0 & \sqrt{3} & 0\\ 0 & 0 & 0 & 2\end{pmatrix}.$$

5. Assemble the square root of X: Define

$$Y=P\,D^{\frac{1}{2}}P^{-1}.$$

Then

$$Y^2=PD^{\frac{1}{2}}\underbrace{P^{-1}P}_{=I}D^{\frac{1}{2}}P^{-1}=P\,D^{\frac{1}{2}}D^{\frac{1}{2}}P^{-1}=PDP^{-1}=X,$$

so $Y=X^{\frac{1}{2}}$. Carrying out the multiplication $PD^{\frac{1}{2}}P^{-1}$ symbolically gives the exact result

ROMANIAN MATHEMATICAL MAGAZINE

$$X^{\frac{1}{2}}=\begin{bmatrix} \frac{\sqrt{3}}{14}+\frac{8\sqrt{2}}{7} & -\frac{9\sqrt{2}}{7}-\frac{\sqrt{3}}{7}+\frac{5}{2} & -1-\frac{3\sqrt{3}}{14}+\frac{4\sqrt{2}}{7} & -\frac{1}{2}-\frac{\sqrt{2}}{7}+\frac{3\sqrt{3}}{7} \\ -\frac{3}{7}+\frac{\sqrt{3}}{14}+\frac{4\sqrt{2}}{7} & -\frac{9\sqrt{2}}{14}-\frac{\sqrt{3}}{7}+\frac{20}{7} & -\frac{5}{7}-\frac{3\sqrt{3}}{14}+\frac{2\sqrt{2}}{7} & -\frac{4}{7}-\frac{\sqrt{2}}{14}+\frac{3\sqrt{3}}{7} \\ -\frac{8}{7}+\frac{\sqrt{3}}{14}+\frac{8\sqrt{2}}{7} & -\frac{9\sqrt{2}}{7}-\frac{\sqrt{3}}{7}+\frac{24}{14} & -\frac{3\sqrt{3}}{14}+\frac{3}{7}+\frac{4\sqrt{2}}{7} & -\frac{5}{14}-\frac{\sqrt{2}}{7}+\frac{3\sqrt{3}}{7} \\ -\frac{5}{7}+\frac{3\sqrt{3}}{14}+\frac{4\sqrt{2}}{7} & -\frac{9\sqrt{2}}{14}-\frac{3\sqrt{3}}{7}+\frac{10}{7} & -\frac{9\sqrt{3}}{14}+\frac{1}{7}+\frac{2\sqrt{2}}{7} & -\frac{2}{7}-\frac{\sqrt{2}}{14}+\frac{9\sqrt{3}}{7} \end{bmatrix}.$$

It can be approximated numerically, to obtain

$$X^{\frac{1}{2}}\approx\begin{pmatrix} 1.73996 & 0.43429 & -0.56303 & 0.04028 \\ 0.50327 & 1.70057 & -0.68138 & 0.06986 \\ 0.59711 & -0.28000 & 0.86554 & 0.18313 \\ 0.46499 & -0.22287 & -0.56654 & 1.84019 \end{pmatrix}.$$

1. Summary

Given only the numerical matrix X, the procedure is:

1. **Compute the characteristic polynomial and its roots (the eigenvalues).**
2. **If all eigenvalues are real and positive (and X is diagonalizable), a real square root exist.**
3. **Find an eigenvector for each eigenvalue and assemble them as columns of P.**
4. **Form $D^{\frac{1}{2}}$ by taking positive square roots of the eigenvalues.**
5. **The square root is $X^{\frac{1}{2}}=PD^{\frac{1}{2}}P^{-1}$.**

This is precisely the general diagonalization method for matrix square roots, applicable whenever the matrix is diagonalizable with a non-negative spectrum.

Solution 2 by Hikmat Mammadov-Azerbaijan

$$X=\begin{pmatrix} 41 & 23 & -25 & 1 \\ 19 & 46 & -29 & 2 \\ 21 & -7 & 7 & 7 \\ 17 & -6 & -23 & 46 \end{pmatrix}$$

$$A-14X=\begin{pmatrix} 41 & 23 & -25 & 1 \\ 19 & 46 & 29 & 2 \\ 21 & -7 & 7 & 7 \\ 17 & -6 & -23 & 46 \end{pmatrix}$$

ROMANIAN MATHEMATICAL MAGAZINE

$$\det(A-\lambda I)=\begin{vmatrix}41-\lambda & 23 & -25 & 1\\ 19 & 46-\lambda & -29 & 2\\ 21 & 7 & 7-\lambda & 7\\ 17 & -6 & -23 & 46-\lambda\end{vmatrix}=0$$

$$\Rightarrow R_1\to R_1-R_4,\qquad R_2\to R_2-R_3$$

$$\det(A-\lambda I)=\begin{vmatrix}24-\lambda & 29 & -2 & -(45-\lambda)\\ -2 & 53-\lambda & -(36-\lambda) & -5\\ 21 & -7 & 7-\lambda & 7\\ 17 & -6 & -23 & 46-\lambda\end{vmatrix}=0$$

$$\Rightarrow tr(A)=41+46+7+46=140$$

$$\Rightarrow tr(A^2)=1610+2744-434+1960=5880$$

$$\Rightarrow P_A(\lambda)=(\lambda-14)(\lambda-28)(\lambda-42)(\lambda-56)=0$$

$$\Rightarrow \sigma(A)=\{14,28,42,56\}\Rightarrow \sigma(X)=\left\{\frac{14}{14},\frac{28}{14},\frac{42}{14},\frac{56}{14}\right\}=\{1,2,3,4\}$$

$$Y^2=X\Rightarrow \sigma(Y)=\left\{\sqrt{1},\sqrt{2},\sqrt{3},\sqrt{4}\right\}=\left\{1,\sqrt{2},\sqrt{3},2\right\}$$

$$p(\lambda)=a_3\lambda^3+a_2\lambda^2+a_1\lambda+a_0$$

$$\begin{cases}p(1)=a_3(1)^3+a_2(1)^2+a_1(1)+a_0=1\\ p(2)=a_3(2)^3+a_2(2)^2+a_1(2)+a_0=\sqrt{2}\\ p(3)=a_3(3)^3+a_2(3)^2+a_1(3)+a_0=\sqrt{3}\\ p(4)=a_3(4)^3+a_2(4)^2+a_1(4)+a_0=2\end{cases}\Rightarrow$$

$$\Rightarrow\begin{pmatrix}1 & 1 & 1 & 1\\ 8 & 4 & 2 & 1\\ 27 & 9 & 3 & 1\\ 64 & 16 & 4 & 1\end{pmatrix}\begin{pmatrix}a_3\\ a_2\\ a_1\\ a_0\end{pmatrix}-\begin{pmatrix}1\\ \sqrt{2}\\ \sqrt{3}\\ 2\end{pmatrix}$$

$$L_1(\lambda)=\frac{(\lambda-2)(\lambda-3)(\lambda-4)}{(1-2)(1-3)(1-4)}=-\frac{1}{6}(\lambda^3-9\lambda^2+26\lambda-24)$$

$$L_2(\lambda)=\frac{(\lambda-1)(\lambda-3)(\lambda-4)}{(2-1)(2-3)(2-4)}=\frac{1}{2}(\lambda^3-8\lambda^2+19\lambda-12)$$

$$L_3(\lambda)=\frac{(\lambda-1)(\lambda-2)(\lambda-4)}{(3-1)(3-2)(3-4)}=-\frac{1}{2}(\lambda^3-7\lambda^2+14\lambda-8)$$

$$L_4(\lambda)=\frac{(\lambda-1)(\lambda-2)(\lambda-3)}{(4-1)(4-2)(4-3)}=\frac{1}{6}(\lambda^3-6\lambda^2+11-6)$$

$$a_3=\frac{1+3\sqrt{2}-3\sqrt{3}}{6},\quad a_2=\frac{-1-8\sqrt{2}+7\sqrt{3}}{2}$$

$$a_1=\frac{-2+57\sqrt{2}-42\sqrt{3}}{6},\ \ a_0=2-6\sqrt{2}+4\sqrt{3}$$

$$\Rightarrow \sqrt{X}=a_3X^3+a_2X^2+a_1X+a_0I$$

Solution 3 by Hikmat Mammadov-Azerbaijana

$$X=\frac{1}{14}\begin{pmatrix}41 & 23 & -25 & 1\\ 19 & 46 & -29 & 2\\ 21 & -7 & 7 & 7\\ 17 & -6 & -23 & 46\end{pmatrix},\ A=14X=\begin{pmatrix}41 & 23 & -25 & 1\\ 19 & 46 & -29 & 2\\ 21 & -7 & 7 & 7\\ 17 & -6 & -23 & 46\end{pmatrix}$$

$$\det(A-\lambda I)=\begin{vmatrix}41-\lambda & 23 & -25 & 1\\ 19 & 46-\lambda & -29 & 2\\ 21 & -7 & 7-\lambda & 7\\ 17 & -6 & -23 & 46-\lambda\end{vmatrix}=0$$

$$\Rightarrow R_1\to R_1-R_4,\ \ R_2\to R_2-R_3$$

$$\det(A-\lambda I)=\begin{vmatrix}24-\lambda & 29 & -2 & -(45-\lambda)\\ -2 & 53-\lambda & -(36-\lambda) & -5\\ 21 & -7 & 7-\lambda & 7\\ 17 & -6 & -23 & 46-\lambda\end{vmatrix}=0$$

$$\Rightarrow tr(A)=41+46+7+46=140$$

$$\Rightarrow tr\,(A^2)=1610+2744-434+1960=5880$$

$$\Rightarrow P_A(\lambda)=(\lambda-14)(\lambda-28)(\lambda-42)(\lambda-56)=0$$

$$\Rightarrow \sigma(A)=\{14,28,42,56\}\Rightarrow \sigma(X)=\{\mu_1,\mu_2,\mu_3,\mu_4\}=\{1,2,3,4\}$$

$$Y^2=X\Rightarrow \sigma(Y)=\{\sqrt{1},\sqrt{2},\sqrt{3},\sqrt{4}\}=\{1,\sqrt{2},\sqrt{3},2\}$$

$$\Rightarrow Y=\sqrt{x}=\sum_{k=1}^{4}\sqrt{\mu_k}\,P_k=1\cdot P_1+\sqrt{2}\cdot P_2+\sqrt{3}\cdot P_3+2\cdot P_4$$

$$P_k=\prod_{j\neq k}\frac{X-\mu_j I}{\mu_k-\mu_j}\ \ (k=1,2,3,4)$$

$$P_1=\frac{(X-2I)(X-3I)(X-4I)}{(1-2)(1-3)(1-4)}=-\frac{1}{6}(X^3-9X^2+26X-24I)$$

$$P_2=\frac{(X-I)(X-3I)(X-4I)}{(2-1)(2-3)(2-4)}=\frac{1}{2}(X^3-8X^2+19X-12I)$$

$$P_3=\frac{(X-I)(X-2I)(X-4I)}{(3-1)(3-2)(3-4)}=-\frac{1}{2}(X^3-7X^2+14X-8I)$$

ROMANIAN MATHEMATICAL MAGAZINE

$$P_4=\frac{(X-I)(X-2I)(X-3I)}{(4-1)(4-2)(4-3)}=\frac{1}{6}(X^3-6X^2+11X-6I)$$

$$\Rightarrow \sqrt{X}=1\cdot P_1+\sqrt{2}\cdot P_2+\sqrt{3}\cdot P_3+2\cdot P_4$$

$$a_3=-\frac{1}{6}(1)+\frac{1}{2}(\sqrt{2})-\frac{1}{2}(\sqrt{3})+\frac{1}{6}(2)=\frac{1+3\sqrt{2}-3\sqrt{3}}{6}$$

$$a_2=-\frac{1}{6}(-9)+\frac{1}{2}(-8\sqrt{2})-\frac{1}{2}(-7\sqrt{3})+\frac{1}{6}(-12)=\frac{-1-8\sqrt{2}+7\sqrt{3}}{2}$$

$$a_1=-\frac{1}{6}(26)+\frac{1}{2}(19\sqrt{2})-\frac{1}{2}(14\sqrt{3})+\frac{1}{6}(22)=\frac{-2+57\sqrt{2}-42\sqrt{3}}{6}$$

$$a_0=-\frac{1}{6}(-24)+\frac{1}{2}(-12\sqrt{2})-\frac{1}{2}(-8\sqrt{3})+\frac{1}{6}(-12)=2-6\sqrt{2}+4\sqrt{3}$$

$$\Rightarrow \sqrt{X}=a_3X^3+a_2X^2+a_1X+a_0I$$