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**UP.637 Find the general solution of the system**

$$x' = x + 2y + 3z,$$

$$y' = 3x + y + 2z,$$

$$z' = 2x + 3y + z.$$

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## *Solution by proposer*

Writing $X = (x, y, z)^T$, the system is $X' = AX$ with

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}.$$

Notice that $A$ is a circulant matrix, each row is a cyclic shift of the previous one. This structure will simplify finding eigenvalues and eigenvectors. Its characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0.$$

Expanding along the first row, we have

$$\det(A - \lambda I) = (1-\lambda)[(1-\lambda)^2 - 6] - 2[3(1-\lambda) - 4] + 3[9 - 2(1-\lambda)]$$

$$= (1-\lambda)(\lambda^2 - 2\lambda - 5) - 2(-3\lambda - 1) + 3(2\lambda + 7).$$

Expanding $(1-\lambda)(\lambda^2 - 2\lambda - 5) = -\lambda^3 + 3\lambda^2 + 3\lambda - 5$, we get

$$\det(A - \lambda I) = -\lambda^3 + 3\lambda^2 + 3\lambda - 5 + 6\lambda + 2 + 6\lambda + 21 = -\lambda^3 + 3\lambda^2 + 15\lambda + 18.$$

Setting this to zero and multiplying by $-1$, we obtain

$$\lambda^3 - 3\lambda^2 - 15\lambda - 18 = 0.$$

Factoring, yields

$$\lambda^3 - 3\lambda^2 - 15\lambda - 18 = (\lambda - 6)(\lambda^2 + 3\lambda + 3),$$

which roots are

$$\lambda_1 = 6, \quad \lambda_{2,3} = -\frac{3}{2} \pm i\frac{\sqrt{3}}{2}.$$

Now, we compute the corresponding eigenvectors:

1. Eigenvector for $\lambda_1 = 6$. Since every row of $A$ sums to 6, the vector $(1, 1, 1)^T$ satisfies $A(1, 1, 1)^T = 6(1, 1, 1)^T$, and $v_1 = (1, 1, 1)^T$ is the eigenvector for $\lambda_1 = 6$.

**2. Eigenvector for $\lambda = -\frac{3}{2} + i\frac{\sqrt{3}}{2}$. Using the circulant structure, with $\omega = e^{\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, the eigenvector associated with this eigenvalue is**

$$v = (1, \omega^2, \omega) = \left(1, -\frac{1}{2} - i\frac{\sqrt{3}}{2}, -\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = p + i,$$

**where**

$$p = \left(1, -\frac{1}{2}, -\frac{1}{2}\right), \quad q = \left(0, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right),$$

**It can be easily checked that $Av = \lambda v$.**

**Now, we write the real solutions from the complex eigen pair.**

**For a complex eigenvalue $\lambda = \alpha + i\beta$ with eigenvector $v = p + iq$, two independent real solutions are given by**

$$u_1(t) = e^{\alpha t}[p\cos\beta t - q\sin\beta t], \qquad u_2(t) = e^{\alpha t}[p\sin\beta t + q\cos\beta t].$$

**Here $\alpha = -\frac{3}{2}, \beta = \frac{\sqrt{3}}{2}$. Define**

$$P(t) = e^{-\frac{3t}{2}}\left(C_2\cos\frac{\sqrt{3}}{2}t + C_3\sin\frac{\sqrt{3}}{2}t\right),$$

$$Q(t) = e^{-\frac{3t}{2}}\left(C_2\sin\frac{\sqrt{3}}{2}t - C_3\cos\frac{\sqrt{3}}{2}t\right),$$

**so that the contribution $C_2u_1 + C_2u_2$ has**

- **First component $(p_x = 1, q_x = 0)$: equal to $P(t)$.**
- **Second component $\left(p_y = -\frac{1}{2}, q_y = -\frac{\sqrt{3}}{2}\right)$:**

$$-\frac{1}{2}(C_2\cos\beta t + C_3\sin\beta t) + \frac{\sqrt{3}}{2}(C_2\sin\beta T - C_3\cos\beta t)\cdot e^{-\frac{3t}{2}} = -\frac{1}{2}P(t) + \frac{\sqrt{3}}{2}Q(t).$$

- **Third component $\left(p_z = -\frac{1}{2}, q_z = \frac{\sqrt{3}}{2}\right)$: $-\frac{1}{2}P(t) - \frac{\sqrt{3}}{2}Q(t)$.**

**Combining the solution from $\lambda_1 = 6$ (along $v_1 = (1,1,1)^T$) with the two real solutions above, we obtain the general solution:**

$$x(t) = C_1e^{6t} + P(t),$$

$$y(t) = C_1e^{6t} - \frac{1}{2}P(t) + \frac{\sqrt{3}}{2}Q(t),$$

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$$z(t) = C_1 e^{6t} - \frac{1}{2}P(t) - \frac{\sqrt{3}}{2}Q(t),$$

where

$$P(t) = e^{-\frac{3t}{2}}\left(C_2 \cos\frac{\sqrt{3}}{2}t + C_3 \sin\frac{\sqrt{3}}{2}t\right),$$

$$Q(t) = e^{-\frac{3t}{2}}\left(C_2 \sin\frac{\sqrt{3}}{2}t - C_3 \cos\frac{\sqrt{3}}{2}t\right),$$

and $C_1, C_2, C_3 \in \mathbb{R}$ are arbitrary constants.

Remark. Adding the three equations gives $x + y + z = 3C_1 e^{6t}$.

The sum grows (or decays) purely along the eigen direction $(1, 1, 1)^T$, reflecting the symmetry of the circulant matrix $A$. The components $y(t)$ and $z(t)$ are mirror images about the shared term $-\frac{1}{2}P(t)$, a direct consequence of the same cyclic symmetry.