ROMANIAN MATHEMATICAL MAGAZINE

UP.636 Solve the linear differential equation

$$y^{IV} - 5y''' + 10y'' - 10y' + 4y = x^4 - 5x^3 + 10x^2 - 10x + 4.$$

Proposed by José Luis Díaz – Barrero – Spain

Solution by proposer

We first solve the associated homogeneous equation

$$y^{IV} - 5y''' + 10y'' - 10y' + 4y = 0.$$

Its characteristic polynomial is $r^4 - 5r^3 + 10r^2 - 10r + 4 = 0$.

This polynomial equation factors as $(r-1)(r-2)(r-1-i)(r-1+i) = 0$,

and has the roots $r_1 = 1, r_2 = 2, r_{3,4} = 1 \pm i$. We have two distinct real roots and one pair of complex conjugate roots (with real part $\alpha = 1$ and imaginary part $\beta = 1$). Then the homogeneous solution $y_h(x)$ is given by

$$y_h(x) = C_1 e^x + C_2 e^{2x} + e^x(C_3 \cos x + C_4 \sin x),$$

where C_1, C_2, C_3, C_4 are arbitrary constants.

Now, we find a particular solution. The right-hand side is a polynomial of degree 4:

$$Q(x) = x^4 - 5x^3 + 10x^2 - 10x + 4.$$

Since $r = 0$ is not a root of the characteristic equation (the constant term is $4 \neq 0$), we look for a particular solution of the same polynomial form

$$y_p(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E.$$

Computing its derivatives, we obtain

$$y_p' = 4Ax^3 + 3Bx^2 + 2Cx + D, y_p'' = 12Ax^2 + 6Bx + 2C,$$

$$y_p''' = 24Ax + 6B, y_p^{IV} = 24A.$$

Substituting into $y^{IV} - 5y''' + 10y'' - 10y' + 4y$, and collecting terms by powers of x, we obtain

$$4Ax^4 + (-40A + 4B)x^3 + (120A - 30B + 4C)x^2$$
$$+(-120A + 60B - 20C + 4D)x + (24A - 30B + 20C - 10D + 4E).$$

Equating with $x^4 - 5x^3 + 10x^2 - 10x + 4$, we have

$$\begin{aligned} x^4 &: 4A &&= 1 \\ x^3 &: -40A + 4B &&= -5 \\ x^2 &: 120A - 30B + 4C &&= 10 \end{aligned}$$

$$x^1: -120A + 60B - 20C + 4D = -10$$
$$x^0: 24A - 30B + 20C - 10D + 4E = 4$$

From the first equation, we get $A = \frac{1}{4}$. Substituting into the second we have

$-10 + 4B = -5 \Rightarrow B = \frac{5}{4}$.

Substituting A, B into the third yields $30 - \frac{75}{2} + 4C = 10 \Rightarrow C = \frac{35}{8}$.

Substituting A, B, C into the fourth:

$$-30 + 75 - \frac{175}{2} + 4D = -10 \Rightarrow D = \frac{65}{8}$$

Substituting A, B, C, D into the fifth:

$$6 - \frac{75}{2} + \frac{175}{2} - \frac{325}{4} + 4E = 4 \Rightarrow E = \frac{117}{16}.$$

Thus, a particular solution is

$$y_p(x) = \frac{1}{4}x^4 + \frac{5}{4}x^3 + \frac{35}{8}x^2 + \frac{65}{8}x + \frac{117}{16}.$$

Combining the homogeneous and particular solutions, $y = y_h + y_p$, we obtain the general solution

$$y(x) = C_1 e^x + C_2 e^{2x} + e^x(C_3 \cos x + C_4 \sin x)$$
$$+ \frac{1}{4}x^4 + \frac{5}{4}x^3 + \frac{35}{8}x^2 + \frac{65}{8}x + \frac{117}{16}$$

where $C_1, C_2, C_3, C_4 \in \mathbb{R}$ are arbitrary constants, determined by initial or boundary conditions if specified.