ROMANIAN MATHEMATICAL MAGAZINE

UP.635 Prove that:

$$\int_0^{\frac{\pi}{2}} (\cos x)^2 e^{\cos(\tan(x))} \cos(\sin(\tan(x)))\, dx = \frac{\pi(e+1)}{4e}\sqrt{e}$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\int_0^{\frac{\pi}{2}} (\cos x)^2 e^{\cos(\tan(x))} \cos(\sin(\tan(x)))\, dx \overset{\tan(x)\to x}{=} \int_0^{\infty} \frac{1}{1+x^2}\cdot e^{\cos(x)}\cdot\cos(\sin(x))\cdot\frac{dx}{1+x^2}$$

$$\Rightarrow \int_0^{\infty} \frac{e^{\cos(x)}\cdot\cos(\sin(x))}{(1+x^2)^2}dx = \Re\left(\int_0^{\infty}\frac{e^{\cos(x)}e^{i\sin(x)}}{(1+x^2)^2}dx\right) = \Re\left(\int_0^{\infty}\frac{e^{e^{ix}}}{(1+x^2)^2}dx\right)$$

$$= \Re\left(\int_0^{\infty}\frac{1}{(1+x^2)^2}\left(\sum_{n=0}^{\infty}\frac{(e^{ix})^n}{n!}\right)dx\right)$$

$$\Rightarrow \int_0^{\infty}\frac{1}{(1+x^2)^2}\left(\sum_{n=0}^{\infty}\frac{\cos(nx)}{n!}\right)dx = \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^{\infty}\frac{\cos(nx)}{(1+x^2)^2}dx$$

$$f(k) = \int_0^{\infty}\frac{\cos(nx)}{k^2+x^2}dx = \frac{\pi}{2k}e^{-kx}$$

$$f'(k) = \int_0^{\infty}\frac{-2k\cos(nx)}{(k^2+x^2)^2}dx = \frac{\pi}{2}\left(-\frac{1}{k^2e^{kn}} - \frac{n}{ke^{kn}}\right)$$

$$\int_0^{\infty}\frac{\cos(nx)}{(1+x^2)^2}dx = \frac{\pi}{4}(1+n)e^{-n}$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^{\infty}\frac{\cos(nx)}{(1+x^2)^2}dx = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{\pi}{4}(1+n)e^{-n}\right] = \frac{\pi}{4}\sum_{n=0}^{\infty}\left[\frac{n+1}{n!}(e^{-1})^n\right]$$

$$\Rightarrow \frac{\pi}{4}\sum_{n=0}^{\infty}\left[\frac{n+1}{n!}(e^{-1})^n\right] = \frac{\pi}{4}\left[\sum_{n=0}^{\infty}\frac{(e^{-1})^n}{n!} + e^{-1}\sum_{n=1}^{\infty}\frac{(e^{-1})^{n-1}}{(n-1)!}\right] =$$

$$= \frac{\pi}{4}\left[e^{e^{-1}} + e^{-1}e^{e^{-1}}\right] = \frac{\pi(e+1)}{4e}\sqrt{e}$$

