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UP.634 Calculate the integral:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \operatorname{arccot} x \, (|\sin x|)^{\frac{1}{3}} (\cos x)^{\frac{5}{3}} \, dx$$

In this problem we will consider that definition of the function $\operatorname{arccot}(x)$ which has the image the interval $(0, \pi)$.

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Solution by proposer

We use the following definition of Euler's Beta function

$$B(p,q) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2p-1} (\cos\theta)^{2q-1} d\theta \qquad (1)$$

where $p > 0, q > 0$. Or, equivalently:

$$B(p,q) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (|\sin\theta|)^{2p-1} (\cos\theta)^{2q-1} d\theta \qquad (2)$$

We will demonstrate that relations (1) and (2) are equivalent.

We have:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (|\sin\theta|)^{2p-1} (\cos\theta)^{2q-1} d\theta = \int_{-\frac{\pi}{2}}^{0} (|\sin\theta|)^{2p-1} (\cos\theta)^{2q-1} d\theta +$$

$$+ \int_0^{\frac{\pi}{2}} (|\sin\theta|)^{2p-1} (\cos\theta)^{2q-1} d\theta = B_1 + B_2$$

$$B_1 = \int_{-\frac{\pi}{2}}^{0} (|\sin\theta|)^{2p-1} (\cos\theta)^{2q-1} d\theta$$

Let us denote: $\theta = -t, t > 0$

$$B_1 = \int_{\frac{\pi}{2}}^{0} (\sin t)^{2p-1} (\cos t)^{2q-1} (-dt) = \int_0^{\frac{\pi}{2}} (\sin t)^{2p-1} (\cos t)^{2q-1} dt$$

$$B_2 = \int_0^{\frac{\pi}{2}} (|\sin\theta|)^{2p-1} (\cos\theta)^{2q-1} d\theta = \int_0^{\frac{\pi}{2}} (\sin\theta)^{2p-1} (\cos\theta)^{2q-1} d\theta$$

I have demonstrated the statement above.

We consider: $2p-1=\frac{1}{3}; 2q-1=\frac{5}{3}$, so: $p=\frac{2}{3}, q=\frac{4}{3}$

Let us denote:

$$A=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \operatorname{arccot} x\,(|\sin x|)^{\frac{1}{3}}(\cos x)^{\frac{5}{3}}dx, \qquad B=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \arctan x\,(|\sin x|)^{\frac{1}{3}}(\cos x)^{\frac{5}{3}}dx$$

We have:

$$A+B=\frac{\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(|\sin x|)^{\frac{1}{3}}(\cos x)^{\frac{5}{3}}dx$$

But $B=0$ because the function to be integrated is odd.

So:

$$A=\frac{\pi}{2}B\left(\frac{2}{3},\frac{4}{3}\right)$$

We use the known relationship:

$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, where $\Gamma(a)$ is the Euler's Gamma function.

$$B\left(\frac{2}{3},\frac{4}{3}\right)=\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma(2)}$$

We will use the known relations:

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha);\ \ \Gamma(\alpha)\Gamma(1-\alpha)=\frac{\pi}{\sin\pi\alpha}; \Gamma(2)=1$$

We obtain:

$$B\left(\frac{2}{3},\frac{4}{3}\right)=\frac{\Gamma\left(\frac{2}{3}\right)\frac{1}{3}\Gamma\left(\frac{1}{3}\right)}{1}=\frac{1}{3}\frac{\pi}{\sin\frac{\pi}{3}}=\frac{2\pi\sqrt{3}}{9}$$

But:

$$A=\frac{\pi}{2}B\left(\frac{2}{3},\frac{4}{3}\right), \qquad A=\frac{\pi^2\sqrt{3}}{9}$$

We obtained the value of the integral A: Thus, the problem is solved.