ROMANIAN MATHEMATICAL MAGAZINE

UP.633 Prove the equality:

$$\int_0^{\infty} \frac{\arctan x \cdot |\ln x|}{x^2 - x + 1} dx = \frac{\pi}{30}\left[\psi_1\left(\frac{1}{6}\right) - 2\pi^2\right]$$

where $\psi_1(x)$ is trigamma function.

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

Let us denote:

$$I = \int_0^{\infty} \frac{\arctan x \cdot |\ln x|}{x^2 - x + 1} dx$$

We also consider the integral:

$$J = \int_0^{\infty} \frac{\arctan x \cdot |\ln x|}{x^2 - x + 1} dx$$

We have

$$I + J = \frac{\pi}{2} \int_0^{\infty} \frac{|\ln x|}{x^2 - x + 1} dx$$

In the integral J we make the change of variable $x = \frac{1}{t}$ and immediately obtain that $J = I$.

Therefore:

$$I = \frac{\pi}{4} \int_0^{\infty} \frac{|\ln x|}{x^2 - x + 1} dx$$

We will calculate:

$$K = \int_0^{\infty} \frac{|\ln x|}{x^2 - x + 1} dx = \int_0^{1} \frac{|\ln x|}{x^2 - x + 1} dx + \int_1^{\infty} \frac{|\ln x|}{x^2 - x + 1} dx$$

Let us denote:

$$A = \int_0^{1} \frac{|\ln x|}{x^2 - x + 1} dx; \quad B = \int_1^{\infty} \frac{|\ln x|}{x^2 - x + 1} dx$$

We can write:

$$A = -\int_0^{1} \frac{(1 + x)\ln x}{1 + x^3} dx$$

We note:

ROMANIAN MATHEMATICAL MAGAZINE

$$C=\int_0^1 \frac{(1+x)\ln x}{1+x^3}dx=\int_0^1 \frac{\ln x}{1+x^3}dx+\int_0^1 \frac{x\ln x}{1+x^3}dx$$

We will calculate:

$$D=\int_0^1 \frac{\ln x}{1+x^3}dx$$

We make the change of variable:

$$y=x^3; x=y^{\frac{1}{3}}$$

We obtain:

$$D=\frac{1}{9}\int_0^1 \frac{y^{-\frac{2}{3}}\ln y}{1+y}dy$$

We will use the following known relationship:

$$\int_0^1 \frac{z^a\ln z}{1+z}dz=\frac{1}{4}\left[\psi_1\left(\frac{a+2}{2}\right)-\psi_1\left(\frac{a+1}{2}\right)\right], a\in\mathbb{R}\setminus\{-1,-2,\dots\}$$

where $\psi_1(x)$ is the trigamma function.

We can therefore write:

$$D=\frac{1}{36}\left[\psi_1\left(\frac{4}{6}\right)-\psi_1\left(\frac{1}{6}\right)\right]$$

By proceeding similarly to the integral D, we obtain:

$$E=\int_0^1 \frac{x\ln x}{1+x^3}dx\frac{1}{36}\left[\psi_1\left(\frac{5}{6}\right)-\psi_1\left(\frac{2}{6}\right)\right]$$

We obtained the value of the integral C:

$$C=D+E=\frac{1}{36}\left[\psi_1\left(\frac{4}{6}\right)-\psi_1\left(\frac{1}{6}\right)+\psi_1\left(\frac{5}{6}\right)-\psi_1\left(\frac{2}{6}\right)\right]$$

So:

$$A=\frac{1}{36}\left[-\psi_1\left(\frac{4}{6}\right)+\psi_1\left(\frac{1}{6}\right)-\psi_1\left(\frac{5}{6}\right)+\psi_1\left(\frac{2}{6}\right)\right]$$

We consider the integral B. We make the variable change: $x=\frac{1}{t}$. We obtain: $B=A$

Result:

$$K=A+B=2A$$

$$K=\frac{1}{18}\left[-\psi_1\left(\frac{4}{6}\right)+\psi_1\left(\frac{1}{6}\right)-\psi_1\left(\frac{5}{6}\right)+\psi_1\left(\frac{2}{6}\right)\right]$$

We have:

$$I=\frac{\pi}{4}K$$

$$I=\frac{\pi}{72}\left[-\psi_1\left(\frac{4}{6}\right)+\psi_1\left(\frac{1}{6}\right)-\psi_1\left(\frac{5}{6}\right)+\psi_1\left(\frac{2}{6}\right)\right]$$

We use the reflection formula:

$$\psi_1(x)+\psi_1(1-x)=\frac{\pi^2}{\sin^2(\pi x)}$$

We obtain:

$$\psi_1\left(\frac{1}{6}\right)+\psi_1\left(\frac{5}{6}\right)=\frac{\pi^2}{\sin^2\frac{\pi}{6}}=4\pi^2,\ \ \psi_1\left(\frac{2}{6}\right)+\psi_1\left(\frac{4}{6}\right)=\frac{\pi^2}{\sin^2\frac{2\pi}{6}}=\frac{4\pi^2}{3}$$

We can write:

$$I=\frac{\pi}{36}\left[\psi_1\left(\frac{1}{6}\right)+\psi_1\left(\frac{2}{6}\right)-\frac{8\pi^2}{3}\right]$$

We will now use the duplication formula:

$$4\psi_1(2x)=\psi_1(x)+\psi_1\left(x+\frac{1}{2}\right)$$

We obtain:

$$\psi_1\left(\frac{2}{6}\right)=\frac{1}{5}\psi_1\left(\frac{1}{6}\right)+\frac{4\pi^2}{15}$$

Result:

$$I=\frac{\pi}{30}\left[\psi_1\left(\frac{1}{6}\right)-2\pi^2\right]$$

Thus, the problem is solved.