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UP.631 Prove that:

$$\int_0^1 \frac{(\log x)^2}{x^2-x-1}dx = \frac{2}{2\varphi-1}\left(Li_3(-\varphi) - Li_3\left(\frac{1}{\varphi}\right)\right)$$

$$where\ \varphi = \frac{1+\sqrt{5}}{2}\ is\ the\ Golden\ Ratio.$$

Proposed by Omkumar Rao-India

Solution 1 by proposer

$$x^2-x-1 \Longrightarrow x_1 = \varphi\ , x_2 = -\frac{1}{\varphi}$$

$$Let\ \Psi = \int_0^1 \frac{(\log x)^2}{x^2-x-1}dx$$

$$\int_0^1 \frac{(\log x)^2}{x^2-x-1}dx = \int_0^1 (\log x)^2\left(\frac{1}{\varphi+\frac{1}{\varphi}}\left(\frac{1}{x-\varphi}-\frac{1}{x+\frac{1}{\varphi}}\right)\right)dx$$

$$= \int_0^1 (\log x)^2\left(\frac{1}{2\varphi-1}\left(\frac{1}{x-\varphi}-\frac{1}{x+\frac{1}{\varphi}}\right)\right)dx$$

$$\Psi = \frac{1}{2\varphi-1}\left[\underbrace{\int_0^1 \frac{(\log x)^2}{x-\varphi}dx}_{-2Li_3\left(\frac{1}{\varphi}\right)} - \underbrace{\int_0^1 \frac{(\log x)^2}{x+\frac{1}{\varphi}}dx}_{-2Li_3(-\varphi)}\right] = \frac{2}{2\varphi-1}\left(Li_3(-\varphi) - Li_3\left(\frac{1}{\varphi}\right)\right)$$

$$\boxed{\int_0^1 \frac{(\log x)^2}{x^2-x-1}dx = \frac{2}{2\varphi-1}\left(Li_3(-\varphi) - Li_3\left(\frac{1}{\varphi}\right)\right)}$$

Solution 2 by Said Attaoui-Algeria

We have $\varphi^2 = \varphi+1, \varphi-1 = \varphi^{-1}$ and $\varphi+\varphi^{-1} = 2\varphi-1 = \sqrt{5}$. Hence

$x^2-x-1 = (x-\varphi)(x+\varphi^{-1})$ and

$$\frac{1}{x^2-x-1} = \frac{1}{\sqrt{5}}\left(\frac{1}{x-\varphi}-\frac{1}{x+\varphi^{-1}}\right).$$

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For $z \in \mathbb{C} \setminus [1,\infty)$ define

$$Li_3(z) = \frac{z}{2}\int_0^1 \frac{\ln^2 t}{1-zt}\,dt.$$

For $|z|<1$, expanding $\frac{1}{1-zt}$ as a geometric series and using $\int_0^1 t^{n-1}\ln^2 t\,dt = \frac{2}{n^3}$ gives $\sum_{n\geq 1}\frac{z^n}{n^3}$. The right-hand side is analytic on $\mathbb{C}\setminus[1,\infty)$, so by the identity theorem it is the principal-branch analytic continuation of the trilogarithm. In particular it is valid at $z=\varphi^{-1}$ (inside the unit disc) and at $z=-\varphi$ (on the negative real axis).

Since

$$\frac{1}{x-\varphi} = -\frac{1}{\varphi}\cdot\frac{1}{1-\frac{x}{\varphi}}, \qquad \frac{1}{x+\varphi^{-1}} = \frac{\varphi}{1+\varphi x},$$

the definition with $z=\varphi^{-1}$ and $z=-\varphi$ yields

$$\int_0^1 \frac{\ln^2 x}{x-\varphi}\,dx = -\frac{1}{\varphi}\cdot 2\varphi Li_3(\varphi^{-1}) = -2Li_3(\varphi^{-1}),$$

$$\int_0^1 \frac{\ln^2 x}{x+\varphi^{-1}}\,dx = \varphi\cdot\frac{2}{-\varphi}Li_3(-\varphi) = -2Li_3(-\varphi)$$

Subtracting and dividing by $\sqrt{5}=2\varphi-1$,

$$\int_0^1 \frac{\ln^2 x}{x^2-x-1}\,dx = \frac{2}{2\varphi-1}\left(Li_3(-\varphi) - Li_3\left(\frac{1}{\varphi}\right)\right).$$

Remark. The relations $\varphi^2=\varphi+1$ and $\varphi^{-1}=\varphi-1$ make the poles φ and $-\varphi^{-1}$ conjugates in $\mathbb{Q}(\sqrt{5})$, with difference $\sqrt{5}=2\varphi-1$. This is the source of the prefactor $\frac{2}{2\varphi-1}$.

Solution 3 by Lara Gangler-Hungary

I. Obtaining two simpler integrals *J* and *K*

$$I = \int_0^1 \frac{\ln^2(x)}{x^2-x-1}\,dx = \int_0^1 \frac{\ln^2(x)}{(x-\varphi)\left(x+\frac{1}{\varphi}\right)}\,dx = \frac{1}{\varphi+\frac{1}{\varphi}}\int_0^1\left(\frac{\ln^2(x)}{x-\varphi} - \frac{\ln^2(x)}{x+\frac{1}{\varphi}}\right)dx =$$

$$= \frac{\varphi}{\varphi^2+1}(K-J)$$

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II. Obtaining two integrals J_1 and J_2 from J by splitting the range of integration

$$J=\int_0^1 \frac{\ln^2(x)}{x+\frac{1}{\varphi}}dx=\underbrace{\int_0^\infty \frac{w^2e^{-w}}{e^{-w}+\frac{1}{\varphi}}dw}_{w=-\ln(x)\to dx=-e^w dw}$$

Problem

The denominator has to be transformed into the form $\frac{1}{1+a}$ with $|a|\le 1$ so that it can be expressed as an infinite sum but it can't be done so uniformly over the whole interval

→ the interval is split in two and the denominator slightly changed and is transformed into the form $\frac{1}{1+a}$

$$J=\int_0^\infty \frac{w^2e^{-w}}{e^{-w}+\frac{1}{\varphi}}dw=\int_0^{\ln(\varphi)} \frac{w^2e^{-w}}{e^{-w}+\frac{1}{\varphi}}dw+\int_{\ln(\varphi)}^\infty \frac{w^2e^{-w}}{e^{-w}+\frac{1}{\varphi}}dw=J_1+J_2$$

$$0\le w\le \ln(\varphi)\to 1\le e^w\le \varphi\to \frac{1}{\varphi}\le \frac{1}{\varphi}e^w\le 1$$

→ the denominator can be expressed as an infinite sum in form $1+\frac{1}{\varphi}e^w$

$$\ln(\varphi)\le w\le \infty\to 0\le e^{-w}\le \frac{1}{\varphi}\to 0\le \varphi e^{-w}\le 1$$

→ the denominator can be expressed as an infinite sum in form $1+\varphi e^{-w}$

III. Calculating J_1

$$J_1=\int_0^{\ln(\varphi)} \frac{w^2e^{-w}}{e^{-w}+\frac{1}{\varphi}}dw=\int_0^{\ln(\varphi)} \frac{w^2}{1+\frac{1}{\varphi}e^w}dw=\int_0^{\ln(\varphi)} \frac{w^2}{1+\frac{1}{\varphi}e^w}dw$$

$$=\int_0^{\ln(\varphi)} w^2\sum_{k=0}^\infty e^{wk}\left(-\frac{1}{\varphi}\right)^k dw=\int_0^{\ln(\varphi)} w^2dw+\int_0^{\ln(\varphi)} w^2\sum_{k=1}^\infty e^{wk}\left(-\frac{1}{\varphi}\right)^k dw=$$

$$=\left[\frac{w^3}{3}\right]_0^{\ln(\varphi)}+\sum_{k=1}^\infty \left(-\frac{1}{\varphi}\right)^k\int_0^{\ln(\varphi)} w^2e^{wk}\,dw=$$

$$=\frac{\ln^3(\varphi)}{3}+\sum_{k=1}^\infty \left(-\frac{1}{\varphi}\right)^k\left[\left(\frac{w^2}{k}-\frac{2w}{k^2}+\frac{2}{k^3}\right)\right]_0^{\ln(\varphi)}=$$

$$=\frac{\ln^3(\varphi)}{3}+\sum_{k}^{\infty}\left(-\frac{1}{\varphi}\right)^k\left[\left(\frac{\ln^2(\varphi)}{k}-\frac{2\ln(\varphi)}{k^2}+\frac{2}{k^3}\right)\varphi^k-\frac{2}{k^3}\right]=$$

$$=\frac{\ln^3(\varphi)}{3}+\sum_{k=1}^{\infty}\left(\ln^2(\varphi)\frac{(-1)^k}{k}-2\ln(\varphi)\frac{(-1)^k}{k^2}+2\frac{(-1)^k}{k^3}-2\frac{\left(-\frac{1}{\varphi}\right)^k}{k^3}\right)=$$

$$=\frac{\ln^3(\varphi)}{3}-\ln^2(\varphi)\ln(2)+2\ln(\varphi)\times\frac{\pi^2}{12}-\frac{3}{2}\zeta(3)-2Li_3\left(-\frac{1}{\varphi}\right)=$$

$$=\frac{\ln^3(\varphi)}{3}-\ln^2(\varphi)\ln(2)+\frac{\pi^2\ln(\varphi)}{6}-\frac{3}{2}\zeta(3)-2Li_3\left(-\frac{1}{\varphi}\right)$$

IV. Calculating J_2

$$J_2=\int_{\ln(\varphi)}^{\infty}\frac{w^2e^{-w}}{e^{-w}+\frac{1}{\varphi}}dw=\int_{\ln(\varphi)}^{\infty}\frac{\varphi w^2e^{-w}}{1+\varphi e^{-w}}dw=\int_{\ln(\varphi)}^{\infty}\varphi w^2\,e^{-w}\sum_{k=0}^{\infty}(-\varphi)^k\,e^{-wk}dw$$

$$=\int_{\ln(\varphi)}^{\infty}-w^2\sum_{k=1}^{\infty}(-\varphi)^k\,e^{-wk}dw=-\sum_{k=1}^{\infty}(-\varphi)^k\int_{\ln(\varphi)}^{\infty}w^2e^{-wk}\,dw$$

$$=\sum_{k=1}^{\infty}(-\varphi)^k\left[\left(\frac{w^2}{k}+\frac{2w}{k^2}+\frac{2}{k^3}\right)e^{-wk}\right]_{\ln(\varphi)}^{\infty}$$

$$=\sum_{k=1}^{\infty}(-\varphi)^k\left[0-\left(\frac{\ln^2(\varphi)}{k}+\frac{2\ln(\varphi)}{k^2}+\frac{2}{k^3}\right)\left(\frac{1}{\varphi}\right)^k\right]$$

$$=\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{\ln^2(\varphi)}{k}+\frac{2\ln(\varphi)}{k^2}+\frac{2}{k^3}\right)$$

$$=\sum_{k}^{\infty}\left(\ln^2(\varphi)\frac{(-1)^{k+1}}{k}+2\ln(\varphi)\frac{(-1)^{k+1}}{k^2}+2\frac{(-1)^{k+1}}{k^3}\right)$$

$$=\ln^2(\varphi)\ln(2)+2\ln(\varphi)\times\frac{\pi^2}{12}+2\times\frac{3}{4}\zeta(3)=\ln^2(\varphi)\ln(2)+\frac{\pi^2\ln(\varphi)}{6}+\frac{3}{2}\zeta(3)$$

V. Calculating J

$$J=J_1+J_2=\frac{\ln^3(\varphi)}{3}-\ln^2(\varphi)\ln(2)+\frac{\pi^2\ln(\varphi)}{6}-\frac{3}{2}\zeta(3)-2Li_3\left(-\frac{1}{\varphi}\right)+$$

$$+\ln^2(\varphi)\ln(2)+\frac{\pi^2\ln(\varphi)}{6}+\frac{3}{2}\zeta(3)=$$

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$$= \frac{\ln^3(\varphi)}{3} + \frac{\pi^2 \ln(\varphi)}{3} - 2Li_3\left(-\frac{1}{\varphi}\right) =$$

$$= \frac{\ln^3(\varphi)}{3} + \frac{\pi^2 \ln(\varphi)}{3} - 2\underbrace{\left(Li_3(-\varphi) + \frac{\ln^3(\varphi)}{6} + \frac{\pi^2 \ln(\varphi)}{6}\right)}_{\substack{\textit{Trilogarithm identity,} \\ \textit{see on the Trilogarithm page} \\ \textit{of Wolfram Alpha}}} = -2Li_3(-\varphi)$$

VI. Calculating K

$$K = \int_0^1 \frac{\ln^2(x)}{x-\varphi}dx = \int_0^1 \frac{\ln^2(x)}{x-\varphi}dx = \int_0^1 \frac{\ln^2(x)}{x-\varphi}dx = -\frac{1}{\varphi}\int_0^1 \frac{\ln^2(x)}{1-\frac{x}{\varphi}}dx =$$

$$= -\frac{1}{\varphi}\int_0^1 \ln^2(x)\underbrace{\sum_{k=0}^{\infty}\left(\frac{x}{\varphi}\right)^k}_{0\le\frac{x}{\varphi}\le 1}dx$$

$$= -\frac{1}{\varphi}\sum_{k=0}^{\infty}\left(\frac{1}{\varphi}\right)^k \int_0^1 x^k \ln^2(x)dx = -\frac{1}{\varphi}\sum_{k=0}^{\infty}\left(\frac{1}{\varphi}\right)^k \frac{\Gamma(3)}{(k+1)^3} = -\frac{2\left(\frac{1}{\varphi}\right)^{k+1}}{(k+1)^3} = -2Li_3\left(\frac{1}{\varphi}\right)$$

VII. End Game

$$I = \frac{\varphi}{\varphi^2+1}(K-J) = \frac{\varphi}{\varphi+2}\left(-2LI_3\left(\frac{1}{\varphi}\right) + 2Li_3(-\varphi)\right) =$$

$$= \frac{2\varphi}{\varphi+2}\left(Li_3(-\varphi) - Li_3\left(\frac{1}{\varphi}\right)\right)$$

$$= \frac{2}{1+\frac{2}{\varphi}}\left(Li_3(-\varphi) - Li_3\left(\frac{1}{\varphi}\right)\right) = \frac{2}{2\varphi-1}\left(Li_3(-\varphi) - Li_3\left(\frac{1}{\varphi}\right)\right)$$