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ON THE PROOF OF THE GENERAL APPROACH TO THE MEASURE OF A CANTOR OPEN SET

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ABSTRACT

measure, interval, segment, Cantor's set, middle interval , length

1. INTRODUCTION

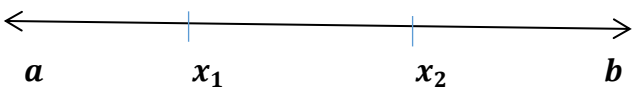
The measure of Cantor's open set G is known to us from a course in real analysis. While the measure is typically considered on the interval [0, 1] we have formulated the problem in a general setting and will proceed to prove it.

2. MAIN PROBLEM AND ITS SOLUTIONS

Problem. Divide the segment [a, b] into 3 equal intervals. Select the middle interval. Then, divide each of the remaining outer intervals into 3 equal intervals. Select the middle intervals obtained from this step, and again divide each of the remaining outer intervals into 3 equal parts; continue this process by repeatedly selecting the middle intervals. Prove that the sum of the lengths of all the middle intervals obtained as a result of this infinite process is equal to the length of the given segment.

Proof:

Process I



Since we have divided it into 3 equal parts, let us set up an equation as follows:

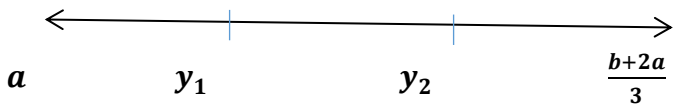
$$x_1 - a = x_2 - x_1 = b - x_2, \qquad x_1 = 2x_2 - b$$

$$2x_2 - b - a = b - x_2, \qquad 3x_2 = 2b + a, \qquad x_2 = \frac{2b+a}{3}$$

$$x_1 = \frac{b+2a}{3}, \qquad \mu_1 = \left(\frac{b+2a}{3}\ ;\ \frac{2b+a}{3}\right), m(\mu_1) = \frac{b-a}{3}$$

Process II

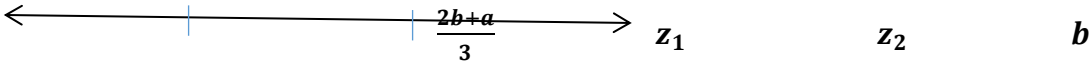
In the second process, the intermediate intervals of the 2 outer intervals obtained in the first process are determined:



$$y_1 - a = y_2 - y_1 = \frac{b+2a}{3} - y_2, \qquad y_1 = 2y_2 - \frac{b+2a}{3}$$

$$2y_2 - \frac{b+2a}{3} - a = \frac{b+2a}{3} - y_2, \qquad 3y_2 = \frac{b+2a}{3} + \frac{b+2a}{3} + a$$

$$y_2 = \frac{7a+2b}{9}, \quad y_1 = \frac{8a+b}{9}, \mu_2 = \left(\frac{8a+b}{9}\ ;\ \frac{7a+2b}{9}\right), m(\mu_2) = \frac{b-a}{9}$$



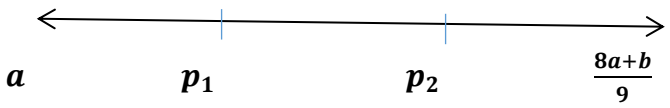
$$z_1 - \frac{2b+a}{3} = z_2 - z_1 = b - z_2, \qquad z_1 = 2z_2 - b$$

$$2z_2 - b - \frac{2b+a}{3} = b - z_2, \qquad z_2 = \frac{8b+a}{9}, \qquad z_1 = \frac{7b+2a}{9}$$

$$\mu_3 = \left(\frac{7b+2a}{9}\ ;\ \frac{8b+a}{9}\right), \qquad m(\mu_3) = \frac{b-a}{9}$$

Process III

In the third process, a middle interval is determined for each of the 4 marginal intervals obtained in the second process:

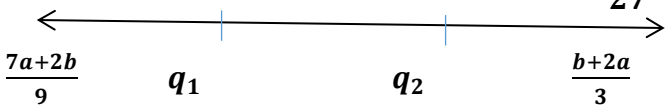


$$p_1 - a = p_2 - p_1 = \frac{8a+b}{9} - p_2, \qquad p_1 = 2p_2 - \frac{8a+b}{9}$$

$$2p_2 - \frac{8a+b}{9} - a = \frac{8a+b}{9} - p_2, \qquad 3p_2 = \frac{25a+2b}{9}$$

$$p_2 = \frac{25a+2b}{27}, \quad p_1 = \frac{26a+b}{27}, \qquad \mu_4 = \left(\frac{26a+b}{27}; \frac{25a+2b}{27}\right)$$

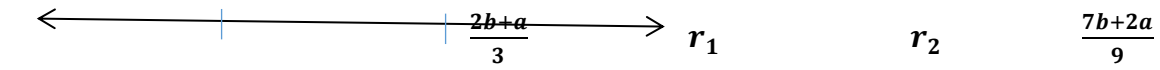
$$m(\mu_4) = \frac{b-a}{27},$$



$$q_1 - \frac{7a+2b}{9} = q_2 - q_1 = \frac{b+2a}{3} - q_2, q_1 = 2q_2 - \frac{b+2a}{3}$$

$$2q_2 - \frac{b+2a}{3} - \frac{7a+2b}{9} = \frac{b+2a}{3} - q_2, 3q_2 = \frac{19a+8b}{9}$$

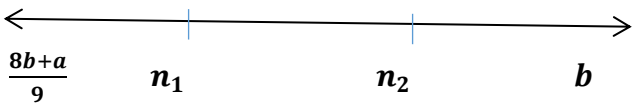
$$q_2 = \frac{19a+8b}{27}, \qquad q_1 = \frac{20a+7b}{27}, \qquad \mu_5 = \left(\frac{20a+7b}{27}; \frac{19a+8b}{27}\right), m(\mu_5) = \frac{b-a}{27}$$



$$r_1 - \frac{2b+a}{3} = r_2 - r_1 = \frac{7b+2a}{9} - r_2, \qquad r_1 = 2r_2 - \frac{7b+2a}{9}$$

$$2r_2 - \frac{7b+2a}{9} - \frac{2b+a}{3} = \frac{7b+2a}{9} - r_2, \qquad 3r_2 = \frac{20b+7a}{9}, \qquad r_2 = \frac{20b+7a}{27}$$

$$r_1 = \frac{19b+8a}{27}, \qquad \mu_6 = \left(\frac{19b+8a}{27}; \frac{20b+7a}{27}\right), \qquad m(\mu_6) = \frac{b-a}{27}$$



$$n_1 - \frac{8b+a}{9} = n_2 - n_1 = b - n_2, \qquad n_1 = 2n_2 - b$$

$$2n_2 - b - \frac{8b+a}{9} = b - n_2, 3n_2 = \frac{26b+a}{9}, n_2 = \frac{26b+a}{27}$$

$$n_1 = \frac{25b+2a}{27}, \mu_7 = \left(\frac{25b+2a}{27}; \frac{26b+a}{27}\right), m(\mu_7) = \frac{b-a}{27}$$

As a result, we obtain an infinite sum based on an infinite process; let us calculate this sum by performing the following technical transformations:

$$\sum_{i=1}^{\infty} m(\mu_i) = m(\mu_1) + m(\mu_2) + m(\mu_3) + m(\mu_4) + m(\mu_5) + m(\mu_6) + m(\mu_7) + \cdots$$

$$= \frac{b-a}{3} + \frac{b-a}{9} + \frac{b-a}{9} + \frac{b-a}{27} + \frac{b-a}{27} + \frac{b-a}{27} + \frac{b-a}{27} =$$

$$= (b-a)\left(\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots\right) = (b-a)\left(\frac{2^0}{3^1} + \frac{2^1}{3^2} + \frac{2^2}{3^3} + \cdots\right) =$$

$$= (b-a)\sum_{j=1}^{\infty} \frac{2^{j-1}}{3^j} = (b-a)2^{-1}\sum_{j=1}^{\infty}\left(\frac{2}{3}\right)^j = \frac{b-a}{2}\frac{\frac{2}{3}}{1-\frac{2}{3}} = b - a = l_{[a,b]}$$

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