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SP.645 Under stereographic projection, what condition must two points z_1, z_2 in the plane satisfy for them to correspond to a pair of antipodal (diametrically opposite) points on the sphere?

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Solution by proposer

Let the sphere be the unit sphere

$$x^2 + y^2 + z^2 = 1,$$

and let the stereographic projection be taken from the north pole $N = (0, 0, 1)$ onto the equatorial plane $z = 0$, which we identify with the complex plane $\mathbb{C}$. For a point $P = (x, y, z)$ on the sphere (different from N), the projection is the point where the line NP meets the plane $z = 0$. That is,

$$\zeta = \frac{x+iy}{1-z} \qquad (1)$$

If $P = (x, y, z)$ has stereographic image $z_1 = \frac{x+iy}{1-z}$, its antipodal point is $P' = (-x, -y, -z)$, whose image is

$$z_2 = \frac{-x - iy}{1 + z}.$$

Now, we find a relation between z_1 and z_2. Computing $z_1\overline{z_2}$, we have

$$\overline{z_2} = \frac{-x + iy}{1 + z},$$

so

$$z_1\overline{z_2} = \frac{(x + iy)(-x + iy)}{(1 - z)(1 + z)} = \frac{-(x^2 + y^2)}{1 - z^2}.$$

Since $x^2 + y^2 = 1 - z^2$ on the sphere, this simplifies to

$$z_1\overline{z_2} = \frac{-(1 - z^2)}{1 - z^2} = -1.$$

Thus, the condition is $z_1\overline{z_2} = -1$ or equivalently $z_2 = -\frac{1}{\overline{z_1}}$.

Geometrically this says:

- $|z_1||z_2| = 1$ – the two points are related by inversion in the unit circle.
- $\arg z_2 = \pi + \arg z_1 \ (mod\ 2\pi)$ – they lie in opposite directions from the origin

So z_1 and z_2 are stereographic projections of antipodal points on the sphere iff $z_1\overline{z_2} = -1$.

Special cases are:

- The projection of the north pole itself is ∞, and its antipode, the south pole, projects to 0.
- If $z_1 = 0$ then $z_2 = \infty$; if z_1 lies on the unit circle ($|z_1| = 1$), then $z_2 = \frac{1}{\overline{z_1}} = -z_1$, its antipode on the same circle.

Finally, we derive the well – known expression (1). Indeed, let $N = (0,0,1)$ be the north pole of the unit sphere $x^2 + y^2 + z^2 = 1$, and let $P = (x,y,z)$ be any point on the sphere other than N. We want the point where the line through N and P meets the equatorial plane

$z = 0$. A point on the line through N and P can be written as

$$Q(t) = N + t(P - N) = (0,0,1) + t\big((x,y,z) - (0,0,1)\big), \quad t \in \mathbb{R}.$$

Writing this out in coordinates:

$$Q(t) = \big(tx, ty, 1 + t(z-1)\big).$$

At $t = 0$ we get N, and at $t = 1$ we get P.

Now, we impose the condition $z = 0$. We seek t such that the third coordinate of $Q(t)$ vanishes:

$$1 + t(z-1) = 0.$$

Since $P \neq N$, we have $z \neq 1$, so

$$t = \frac{1}{1-z}.$$

Substituting into the first two coordinates yields

$$Q\left(\frac{1}{1-z}\right) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right).$$

Identifying $(u,v,0)$ with $u + iv$, the intersection point corresponds to

$$\zeta = \frac{x}{1-z} + i\frac{y}{1-z} = \frac{x+iy}{1-z}.$$