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SP.644 Let a, b, c, d be nonnegative real numbers such that $a \geq b \geq 1 \geq c \geq d$ and $ab + ac + ad + bc + bd + cd = 6$. Prove that

$$\frac{1}{2a+1} + \frac{1}{2b+1} + \frac{1}{2c+1} + \frac{1}{2d+1} \geq \frac{a+b+c+d}{3}$$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

Write the inequality as $E \geq 0$, where

$$E = \frac{1}{2a+1} + \frac{1}{2b+1} + \frac{1}{2c+1} + \frac{1}{2d+1} - \frac{a+b+c+d}{3}.$$

For fixed a and b, assume that c and E are functions of d. We have $c' = \frac{-(a+b+c)}{a+b+d}$ and

$$3E'(d) = -\left[\frac{6}{(2c+1)^2} + 1\right]c' - \frac{6}{(2d+1)^2} - 1 =$$

$$= \frac{(2c+1)^2+6}{(2c+1)^2} \cdot \frac{a+b+c}{a+b+d} - \frac{(2d+1)^2+6}{(2d+1)^2}$$

We claim that $E'(d) \leq 0$. Write this inequality as follows:

$$\frac{a+b+c}{a+b+d} \leq \frac{[(2d+1)^2+6](2c+1)^2}{[(2c+1)^2+6](2d+1)^2}, \quad \frac{a+b+c}{a+b+d} - 1$$

$$\leq \frac{[(2d+1)^2+6](2c+1)^2}{[(2c+1)^2+6](2d+1)^2} - 1,$$

$$\frac{c-d}{a+b+d} \leq \frac{24(c-d)(c+d+1)}{[(2c+1)^2+6](2d+1)^2}.$$

It is true if

$$\frac{1}{a+b+d} \leq \frac{24(c+d+1)}{[(2c+1)^2+6](2d+1)^2}.$$

Since $(2c+1)^2 + 6 \leq 15$ and $a + b + d \geq 2 + d \geq 2d + 1$, it suffices to show that

$$1 \leq \frac{8(c+d+1)}{5(2d+1)},$$

which is equivalent to the obvious inequality $8c - 2d + 3 \geq 0$. From $E'(d) \leq 0$, it follows that $E(d)$ is decreasing and has the minimum value when d is maximum, hence when $d = c$. So, it suffices to consider $d = c$, when we need to show that $F \geq 0$ for

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$$ab + 2(a+b)c + c^2 = 6, \quad a \geq b \geq 1 \geq c,$$

where

$$F = \frac{1}{2a+1} + \frac{1}{2b+1} + \frac{2}{2c+1} - \frac{a+b+2c}{3}.$$

From $6 \geq ab \geq b^2$, we get $b \leq \sqrt{6}$.

For fixed a, assume that c and F are functions of b. We have $2(a+b+c)c' + a + 2c = 0$ and

$$3F'(b) = -\left[\frac{12}{(2c+1)^2} + 2\right]c' - \frac{6}{(2b+1)^2} - 1 =$$

$$= \frac{(2c+1)^2+6}{(2c+1)^2} \cdot \frac{a+2c}{a+b+c} - \frac{(2b+1)^2+6}{(2b+1)^2}.$$

We will show that $F'(b) \geq 0$. Write this inequality as follows:

$$\frac{[(2c+1)^2+6](2b+1)^2}{[(2b+1)^2+6](2c+1)^2} \geq \frac{a+b+c}{a+2c},$$

$$\frac{[(2c+1)^2+6](2b+1)^2}{[(2b+1)^2+6](2c+1)^2} - 1 \geq \frac{a+b+c}{a+2c} - 1,$$

$$\frac{24(b-c)(b+c+1)}{(4b^2+4b+7)(2c+1)^2} \geq \frac{b-c}{a+2c}.$$

It is true if

$$\frac{24(b+c+1)}{(4b^2+4b+7)(2c+1)^2} \geq \frac{1}{a+2c}.$$

Since

$$(b+c+1)(a+2c) \geq (b+2c)(a+2c) \geq (b+2c)^2,$$

it suffices to show that

$$\frac{24(b+2c)^2}{(2c+1)^2} \geq 4b^2 + 4b + 7.$$

Since

$$\frac{b+2c}{2c+1} = 1 + \frac{b-1}{2c+1} \geq 1 + \frac{b-1}{3} = \frac{b+2}{3},$$

we have

$$\frac{24(b+2c)^2}{(2c+1)^2}-(4b^2+4b+7)\geq\frac{8(b+2)^2}{3}-(4b^2+4b+7)=$$

$$=\frac{4(6-b^2)+20b-13}{3}>0$$

From $F'(b)\geq 0$, it follows that $F(b)$ is increasing and has the minimum value when b is minimum, hence when $b=1$. So, it suffices to consider this case, when we need to show that

$$\frac{1}{2a+1}+\frac{1}{3}+\frac{2}{2c+1}\geq\frac{a+1+2c}{3}$$

for $a=\frac{6-2c-c^2}{2c+1}$. Since

$$2a+1=\frac{13-2c-2c^2}{2c+1},\qquad \frac{1}{3}+\frac{2}{2c+1}=\frac{7+2c}{3(2c+1)},\quad a+1+2c=\frac{7+2c+3c^2}{2c+1},$$

the required inequality becomes as follows:

$$\frac{2c+1}{13-2c-2c^2}+\frac{7+2c}{3(2c+1)}\geq\frac{7+2c+3c^2}{3(2c+1)},\qquad \frac{2c+1}{13-2c-2c^2}\geq\frac{c^2}{2c+1},$$

$$2c^4+2c^3-9c^2+4c+1\geq 0,\ \ (c-1)^2(2c^2+6c+1)\geq 0.$$

The proof is finished. The equality occurs for $(a,b,c,d)=(1,1,1,1)$.

Remark. Actually, the following generalization is true:

- **If $k\geq k_0=\frac{7+\sqrt{65}}{8}\approx 1.8828$, then**

$$\frac{1}{ka+1}+\frac{1}{kb+1}+\frac{1}{kc+1}+\frac{1}{kd+1}\geq\frac{a+b+c+d}{k+1}$$

for all nonnegative real numbers a,b,c,d such that $a\geq b\geq 1\geq c\geq d$ and

$$ab+ac+ad+bc+bd+cd=6.$$

If $k=k_0$, then the equality occurs for $(a,b,c,d)=(1,1,1,1)$ and also for $(a,b,c,d)=(6,1,0,0)$.