ROMANIAN MATHEMATICAL MAGAZINE

SP.643 Find the equation of the common perpendicular line and the minimum distance between the lines r and s, given by

$$r:\begin{cases}2x-y-2=0\\ y-2z+6=0\end{cases} \qquad s:\begin{cases}x-y-2=0\\ 2y-z+1=0\end{cases}$$

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Solution by proposer

We have

1. Line r: direction vector via $\overrightarrow{n_1}\times\overrightarrow{n_2}$, where $\overrightarrow{n_1}=(2,-1,0),\overrightarrow{n_2}=(0,1,-2)$:

$$\vec{u}=\overrightarrow{n_1}\times\overrightarrow{n_2}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\2&-1&0\\0&1&-2\end{vmatrix}=(2,4,2)\parallel(1,2,1)$$

Point on r: $A(1,0,3)$. So r: point $A(1,0,3)$, direction $\vec{u}(1,2,1)$.

2. Line s: direction vector via $\overrightarrow{n_1}\times\overrightarrow{n_2}$, where $\overrightarrow{n_1}=(1,-1,0),\overrightarrow{n_2}=(0,2,-1)$:

$$\vec{v}=\overrightarrow{n_1}\times\overrightarrow{n_2}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&-1&0\\0&2&-1\end{vmatrix}=(1,1,2)$$

Point on s: $B(2,0,1)$. So s: point $B(2,0,1)$, direction $\vec{v}(1,1,2)$.

3. Vector between point:

$$\overrightarrow{AB}=B-A=(1,0,-2)$$

4. Cross product $\vec{u}\times\vec{v}$:

$$\vec{u}\times\vec{v}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&2&1\\1&1&2\end{vmatrix}=(3,-1,-1)$$

5. Scalar triple product (check skew lines):

$$\overrightarrow{AB}\cdot(\vec{u}\times\vec{v})=(1)(3)+(0)(-1)+(-2)(-1)=5\neq 0$$

Since this is nonzero, r and s are skew lines. Then, we have

- **Line r: point $A(1,0,3)$, direction $\vec{u}(1,2,1)$**
- **Line s: point $B(2,0,1)$, direction $\vec{v}(1,1,2)$**
- **Common perpendicular direction: $\vec{w}=\vec{u}\times\vec{v}=(3,-1,-1)$**

Next, we find the feet of the common perpendicular. Let $P = A + t\vec{u} = (1 + t, 2t, 3 + t)$ on r, and $Q = B + s\vec{v} = (2 + s, s, 1 + 2s)$ on s. We need $\overrightarrow{PQ} \perp \vec{u}$ and $\overrightarrow{PQ} \perp \vec{v}$, where $\overrightarrow{PQ} = (1 + s - t, s - 2t, -2 + 2s - t)$.

$$\overrightarrow{PQ} \cdot \vec{u} = 0 \Rightarrow 5s - 6t = 1$$

$$\overrightarrow{PQ} \cdot \vec{v} = 0 \Rightarrow 6s - 5t = 3$$

Solving: $s = \frac{13}{11}, t = \frac{9}{11}$.

Now, we compute the feet of the perpendicular:

$$P = \left(\frac{20}{11}, \frac{18}{11}, \frac{42}{11}\right), \quad Q = \left(\frac{35}{11}, \frac{13}{11}, \frac{37}{11}\right)$$

Check: $\overrightarrow{PQ} = \left(\frac{15}{11}, -\frac{5}{11}, -\frac{5}{11}\right) = \frac{5}{11}(3, -1, -1)$ parallel to $\vec{w}$.

Equation of the common perpendicular line. Passes through $P\left(\frac{20}{11}, \frac{18}{11}, \frac{42}{11}\right)$ with direction $\vec{w}(3, -1, -1)$:

$$\frac{x - \frac{20}{11}}{3} = \frac{y - \frac{18}{11}}{-1} = \frac{z - \frac{42}{11}}{-1}$$

- Plane containing r and $\vec{w}$: normal $= \vec{u} \times \vec{w} = (-1, 4, -7)$, through $A(1, 0, 3)$ is $x - 4y + 7z - 22 = 0$.
- Plane containing s and $\vec{w}$: normal $= \vec{v} \times \vec{w} = (1, 7, -4)$, through $B(2, 0, 1)$ is $x + 7y - 4z + 2 = 0$.

Thus, the equation of common perpendicular is:

$$\begin{cases} x - 4y + 7z - 22 = 0 \\ x + 7y - 4z + 2 = 0 \end{cases}$$

Finally, the minimum distance is

$$d(r, s) = \left|\overrightarrow{PQ}\right| = \frac{5}{11}\sqrt{3^2 + (-1)^2 + (-1)^2} = \frac{5}{11}\sqrt{11} = \frac{5\sqrt{11}}{11}.$$