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SP.642 It is possible to write each positive integer as a sum of distinct Lucas numbers such that no two summands are neighbors in the Lucas sequence? The same question for Fibonacci numbers.

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Solution by proposer

(a) Lucas numbers. Recall that $L_0 = 2, L_1 = 1$ and $L_{k+1} = L_k + L_{k-1}$ for $k \geq 1$, so the sequence is $2, 1, 3, 4, 7, 11, 18, 29, \ldots$ Note that $L_1 < L_2 < L_3 < \cdots$, and that

$$L_{k+1} - L_k = L_{k-1} \quad (k \geq 1).$$

We prove by strong induction on n that every positive integer n is a sum of distinct Lucas numbers, no two of them with consecutive indices.

For $n \leq 3$ the statement is immediate: $1 = L_1, 2 = L_0$ and $3 = L_2$ are themselves Lucas numbers, so each of them is a sum with a single summand, and the condition that no two summands are neighbors in the sequence is satisfied trivially.

Let $n \geq 4$ and assume the statement holds for all positive integers less than n. Since $L_3 = 4 \leq n$ and $L_k \to \infty$, there is a unique $k \geq 3$ with

$$L_k \leq n < L_{k+1}$$

Put $m = n - L_k$. Then $0 \leq m < L_{k+1} - L_k = L_{k-1}$. If $m = 0$, then $n = L_k$ and we are done. Otherwise $1 \leq m < n$, and by the induction hypothesis m is a sum of distinct Lucas numbers with no two consecutive indices. Every monotonicity of $L_1, L_2, \ldots$ gives $j \leq k - 2$; if $j = 0$, then j is not adjacent to k because $k \geq 3$. Hence adding the summand L_k to the representation of neighbors in the sequence. This completes the induction.

For instance, $5 = L_2 + L_0 = 3 + 2, 16 = L_5 + L_3 + L_1 = 11 + 4 + 1$ and

$$17 = L_5 + L_3 + L_0 = 11 + 4 + 2.$$

(b) Fibonacci numbers. Let $F_1 = F_2 = 1$ and $F_{k+1} = F_k + F_{k-1}$. Since $F_1 = F_2$, we use only $F_2, F_3, F_4, \ldots = 1, 2, 3, 5, 8, \ldots$, which is strictly increasing, and $F_{k+1} - F_k = F_{k-1}$. The argument is the same. We have $1 = F_2$. For $n \geq 2$, let $k \geq 3$ be such that $F_k \leq n < F_{k+1}$, and put

$m = n - F_k$, so that $0 \leq m < F_{k-1}$. If $m = 0$ we are done; otherwise, by the induction hypothesis m is a sum of distinct, pairwise non-consecutive Fibonacci numbers, all of them

at most $m < F_{k-1}$, hence with indices at most $k-2$. Adding F_k gives the required representation of n.

Thus, every positive integer is a sum of distinct Fibonacci numbers, no two of them consecutive. This is Zeckendorf's theorem.

Remark. The Fibonacci representation is unique. The Lucas representation is not: for example,

$5 = L_2 + L_0 = L_3 + L_1$. Uniqueness is restored if we also forbid using L_0 and L_2 together.