ROMANIAN MATHEMATICAL MAGAZINE

**SP.641** Let $A(z)$ be a polynomial with complex coefficients, and let $a, b, c \in \mathbb{C}$. Suppose that $A(z)$ leaves the same remainder upon division by $(z-a)(z-b)$ and upon division by $(z-a)(z-c)$. Determine the value of

$$(a-b)A(c)+(b-c)A(a)+(c-a)A(b).$$

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*Solution by proposer*

Since both divisors have degree $2$, the common remainder has degree at most $1$. Write is as

$R(z)=pz+q$ with $p, q \in \mathbb{C}$. Then there exist polynomials $Q_1(z)$ and $Q_2(z)$ such that

$$A(z)=(z-a)(z-b)Q_1(z)+pz+q \quad \text{and} \quad A(z)=(z-a)(z-c)Q_2(z)+pz+q.$$

Substituting $z=a$ and $z=b$ in the first identity, and $z=c$ in the second, we obtain

$$A(a)=pa+q,\ \ A(b)=pb+q,\ \ A(c)=pc+q.$$

Hence

$$S=(a-b)(pc+q)+(b-c)(pa+q)+(c-a)(pb+q)$$
$$=p[(a-b)c+(b-c)a+(c-a)b]+q[(a-b)+(b-c)+(c-a)]$$
$$=p(ac-bc+ab-ac+bc-ab)+q\cdot 0=0.$$

Therefore

$$(a-b)A(c)+(b-c)A(a)+(c-a)A(b)=0.$$

Remarks.

1. The expression equals, up to sign, the determinant

$$\begin{vmatrix} a & A(a) & 1 \\ b & A(b) & 1 \\ c & A(c) & 1 \end{vmatrix},$$

which vanishes because the points $(a, A(a)), (b, A(b)), (c, A(c))$ all satisfy the linear relation $w=pz+q$. The hypothesis on the reminder is precisely what forces this "collinearity".

2. The result remains valid when $a, b, c$ are not distinct. For instance, if $b=a$, the first divisor is $(z-a)^2$ and we still have $A(a)=A(b)=R(a)$. If $b=c$, the expression vanishes identically.