ROMANIAN MATHEMATICAL MAGAZINE

SP.640 Find all prime numbers p, q, r with $q < 32$ such that

$$p^q + q^p = r(4p + 1).$$

Proposed by José Luis Díaz – Barrero – Spain

Solution 1 by proposer

We show that the solutions are $(p, q, r) = (2, 11, 241)$ and $(p, q, r) = (2, 13, 929)$. Note that $4p + 1$ is always odd. We distinguish three cases.

Case 1: p and q are both odd. Then $p^q + q^p$ is even while $4p + 1$ is odd, so $r = 2$ and

$$p^q + q^p = 8p + 2$$

But $p \geq 3$ and $q \geq 3$ give $q^p \geq 3^p$, and $3^p > 8p + 2$ for every $p \geq 3$ (it holds for $p = 3$, since $27 > 26$, and the left side more than triples while the right side increases by 8 when p increases by 1). Hence $p^q + q^p > 8p + 2$, a contradiction.

Case 2: $q = 2$ and p is odd. Let $m = 4p + 1$. Then $4p \equiv -1 (mod\ m)$, so $16p^2 \equiv 1 (mod\ m)$. Multiplying the congruence $p^2 + 2^p \equiv 0 (mod\ m)$ by 16 we get

$$2^{p+4} \equiv -1\ (mod\ m).$$

Since $p + 4$ is odd, we can write $2^{p+4} = 2y^2$ with $y = 2^{\frac{p+3}{2}}$. Then $2y^2 \equiv -1\ (mod\ m)$, and multiplying by 2 gives

$$(2y)^2 \equiv -2\ (mod\ m).$$

Hence -2 is q quadratic residue modulo every prime divisor of m (note that m is odd and coprime to $2y$), and therefore the Jacobi symbol satisfies $\left(\frac{-2}{m}\right) = 1$. On the other hand, p is odd, so $p \equiv 1$ or $3\ (mod\ 4)$ and $m = 4p + 1 \equiv 5$ or $13\ (\text{mod } 16)$; in both cases $m \equiv 5\ (\text{mod } 8)$. Therefore

$$\left(\frac{-1}{m}\right) = 1, \quad \left(\frac{2}{m}\right) = -1, \quad \left(\frac{-2}{m}\right) = \left(\frac{-1}{m}\right)\left(\frac{2}{m}\right) = -1,$$

a contradiction. So, there are no solutions in this case.

Case 3: $p = 2$. Now $4p + 1 = 9$ and the equation becomes

$$2^q + q^2 = 9r.$$

For $q = 2$ and $q = 3$ the left side equals 8 and 17, which are not divisible by 9. Let $q \geq 5$.

Since $2^6 \equiv 1 \pmod 9$ and $q \equiv 1$ or $5 \pmod 6$, we have

$$2^q \equiv 2 \text{ if } q \equiv 1 \pmod 6, \quad 2^q \equiv 5 \text{ if } q \equiv 5 \pmod 6.$$

Also $q^2 \equiv 1, 4, 7 \pmod 9$ according as $q \equiv \pm 1, \pm 2, \pm 4 \pmod 9$. Hence $9 | 2^q + q^2$ requires either $q \equiv 1 \pmod 6$ and $q^2 \equiv 7$, that is $q \equiv \pm 4 \pmod 9$, or $q \equiv 5 \pmod 6$ and $q^2 \equiv 4$, that is, $q \equiv \pm 2 \pmod 9$. Combining these conditions gives

$$9 | 2^q + q^2 \Leftrightarrow q \equiv 11 \text{ or } 13 \pmod{18}.$$

The primes $q < 32$ with this property are $11, 13, 29$ and 31.

- $q = 11$: $2^{11} + 11^2 = 2048 + 121 = 2169 = 9 \cdot 241$, and 241 is prime (it is not divisible by $2, 3, 5, 7, 11, 13$ and $17^2 > 241$).
- $q = 13$: $2^{13} + 13^2 = 8192 + 169 = 8361 = 9 \cdot 929$, and 929 is prime (it is not divisible by $2, 3, 5, 7, 11, 13, 17, 19, 23, 29$, and $31^2 > 929$).
- $q = 29$ and $q = 31$: we show that $27 | 2^q + q^2$, so that r is divisible by 3; since $r > 3$, it is not prime. As $2^{18} \equiv 1 \pmod{27}$,

$$2^{29} + 29^2 \equiv 2^{11} + 2^2 = 2048 + 4 = 2052 = 27 \cdot 76 \equiv 0 \pmod{27},$$

$$2^{31} + 31^2 \equiv 2^{13} + 4^2 = 8192 + 16 = 8208 = 27 \cdot 304 \equiv 0 \pmod{27}.$$

Therefore, the solutions are

$$(p, q, r) = (2, 11, 241) \text{ and } (p, q, r) = (2, 13, 929).$$

Solution 2 by Jenish Rijal-Bardiya-Nepal

Case I: q is odd.

If p is odd, then $p^q + q^p$ is even, forcing $r = 2$.

So, $$p^q + q^p = 8p + 2$$

But $$p^q + q^p \geq p^3 > 8p + 2 \quad \text{for } p \geq 3.$$

Thus, $p = 2$ and the equation now becomes:

$$q^2 + 2^q = 9r$$

As $q < 32$ is a prime, mod 9 gives,

q	3	5	7	11	13	17	19	23	29	31
$q^2 + 2^q \pmod 9$	8	3	6	0	0	6	3	3	0	0

Hence, $q \in \{11, 13, 29, 31\}$

Since $2^{18} \equiv 1 \pmod{27}$,

$$29^2 + 2^{29} \equiv 2^2 + 2^{11} \equiv 4 + 23 \equiv 0 \pmod{27}$$

and $$31^2 + 2^{31} \equiv 4^2 + 2^{13} \equiv 16 + 11 \equiv 0 \pmod{27}$$

Thus, for $q = 29$ or 31, we have $3 \mid r$, forcing $r = 3$ which is impossible since $q^2 + 2^q > 27$.

Now, for $q \in \{11, 13\}$, we get,

$$(p, q, r) = (2, 11, 241) \text{ or } (2, 13, 929).$$

as the valid solutions of Case I.

Solution 1 for *Case II*:

Case II: $q = 2$.

The equation now becomes: $p^2 + 2^p = r(4p + 1)$ If $p = 2$, we get $8 = 9r$ (impossible)

Hence p is odd. Let $m = 4p + 1$. We get, $m \mid p^2 + 2^p$

Since $4p \equiv -1 \pmod{m}$,

$$16p^2 \equiv 1 \pmod{m}$$

$$16 \cdot 2^p \equiv -1 \pmod{m}$$

Now $m \equiv 5 \pmod{8}$, so

$$\left(\frac{2}{m}\right) = -1 \text{ and } \left(\frac{-1}{m}\right) = 1$$

Taking Jacobi symbols mod m, we have,

$$1 = \left(\frac{-1}{m}\right) = \left(\frac{16 \cdot 2^p}{m}\right) = \left(\frac{16}{m}\right)\left(\frac{2}{m}\right)^p = (-1)^p = -1, \text{ a contradiction.}$$

Therefore $q \neq 2$, which completes the proof!

Solution 2 for *Case II*:

Case II: $q = 2$.

If $p = 2$, we get $8 = 9r$ (impossible) The equation now becomes: $p^2 + 2^p = r(4p + 1)$

Hence p is odd. For any $p \geq 3$ we have $p^2 \equiv 1 \pmod{8}$ and $4p + 1 \equiv 5 \pmod{8}$.

So, $p^2 + 2^p \equiv 1 + 0 \equiv 1 \pmod{8} \implies 1 \equiv r \cdot 5 \pmod{8} \therefore r \equiv 5 \pmod{8}$

Let $r = 8k + 5$ for $k \geq 0$. Let us consider the following even numbers:

$$2, 4, 6, 8, \ldots\ldots, 2(4k + 2)$$

Taking mod r, $2 \equiv 2 \pmod{r}, 4 \equiv 4 \pmod{r}, 6 \equiv 6 \pmod{r}$....., $4k+2 \equiv 4k+2 \pmod{r}$ and $4k+4 \equiv -(4k+1) \pmod{r}, 4k+6 \equiv -(4k-1) \pmod{r}$, ..., $2(4k+2) \equiv -1 \pmod{r}$

Multiplying all the congruences, we get,

$$2^{4k+2} \cdot (4k+2)! \equiv -(4k+2)! \pmod{r}$$

Since r is a prime number $> 4k+2$, we can cancel $(4k+2)!$ and get,

$$2^{4k+2} \equiv -1 \pmod{r} \Longrightarrow 2^{\frac{r-1}{2}} \equiv -1 \pmod{r}$$

We have, $p^2 \equiv -2^p \pmod{r} \Rightarrow \left(p^2\right)^{\frac{r-1}{2}} \equiv (-2^p)^{\frac{r-1}{2}} \pmod{r} \Rightarrow p^{r-1} \equiv (-1)^{4k+2}\left(2^{4k+2}\right)^p \pmod{r}$

$$\Rightarrow p^{r-1} \equiv 1(-1)^p \equiv -1 \pmod{r}$$

, which is a contradiction because by Fermat's little theorem, $p^{r-1} \equiv 1 \pmod{r}$

$[r \nmid p$ since $p^2 \equiv -2^p \not\equiv 0 \pmod{r}]$

Hence, the required triples are $(p, q, r) = (2, 11, 241)$ or $(2, 13, 929)$.