

ROMANIAN MATHEMATICAL MAGAZINE

SP.639 If $a, b, c > 0$ and $1 \leq \lambda \leq 2$ then:

$$\sum \frac{xy}{x^2 + \lambda y^2} \leq \frac{3}{\lambda + 1}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

$$\begin{aligned} LHS &= \sum \frac{xy}{x^2 + \lambda y^2} = \sum \frac{xy}{x^2 + y^2 + (\lambda - 1)y^2} \leq \sum \frac{xy}{2xy + (\lambda - 1)y^2} = \\ &= \sum \frac{x}{2x + (\lambda - 1)y} = \\ &= \sum \frac{1}{2} \left(1 - \frac{(\lambda - 1)y}{2x + (\lambda - 1)y} \right) = \frac{1}{2} \left(3 - \sum \frac{(\lambda - 1)y}{2x + (\lambda - 1)y} \right) = \\ &= \frac{3}{2} - \frac{1}{2} \sum \frac{(\lambda - 1)y}{2x + (\lambda - 1)y} \leq \frac{3}{\lambda + 1} = RHS, \end{aligned}$$

$$\text{where } \frac{3}{2} - \frac{1}{2} \sum \frac{(\lambda - 1)y}{2x + (\lambda - 1)y} \leq \frac{3}{\lambda + 1} \Leftrightarrow \frac{1}{2} \sum \frac{(\lambda - 1)y}{2x + (\lambda - 1)y} \geq \frac{3(\lambda - 1)}{2(\lambda + 1)} \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{y}{2x + (\lambda - 1)y} \geq \frac{3}{\lambda + 1},$$

$$\text{see } \sum \frac{y}{2x + (\lambda - 1)y} = \sum \frac{y^2}{2xy + (\lambda - 1)y^2} \stackrel{CS}{\geq} \frac{(\sum y)^2}{\sum (2xy + (\lambda - 1)y^2)} = \frac{\sum x^2 + 2 \sum xy}{2 \sum xy + (\lambda - 1) \sum x^2} \geq \frac{3}{\lambda + 1} \text{ where}$$

$$\frac{\sum x^2 + 2 \sum xy}{2 \sum xy + (\lambda - 1) \sum x^2} \geq \frac{3}{\lambda + 1} \Leftrightarrow$$

$$\Leftrightarrow (\lambda + 1) \sum x^2 + 2(\lambda + 1) \sum xy \geq 6 \sum xy + 3(\lambda - 1) \sum x^2 \Leftrightarrow$$

$$\Leftrightarrow (4 - 2\lambda) \sum x^2 \geq (4 - 2\lambda) \sum xy, \text{ see } \lambda \leq 2 \text{ and } \sum x^2 \geq \sum xy.$$

Equality holds if and only if $x = y = z$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum_{cyc} \frac{xy}{x^2 + \lambda y^2} &= \sum_{cyc} \frac{xy}{x^2 + (\lambda - 1)y^2 + y^2} \stackrel{AM-GM}{\leq} \\ \frac{1}{2} \cdot \sum_{cyc} \frac{(2xy + (\lambda - 1)y^2) - (\lambda - 1)y^2}{2xy + (\lambda - 1)y^2} &= \frac{3}{2} - \frac{\lambda - 1}{2} \cdot \sum_{cyc} \frac{y^2}{2xy + (\lambda - 1)y^2} \\ &\stackrel{\text{Bergstrom}}{\leq} \frac{3}{2} - \frac{\lambda - 1}{2} \cdot \frac{(\sum_{cyc} x)^2}{2 \sum_{cyc} xy + (\lambda - 1) \sum_{cyc} x^2} (\because \lambda \geq 1) \stackrel{?}{\leq} \frac{3}{\lambda + 1} \end{aligned}$$

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$$\begin{aligned}
 & \Leftrightarrow \frac{3(\lambda - 1)}{2(\lambda + 1)} \stackrel{?}{\leq} \frac{\lambda - 1}{2} \cdot \frac{(\sum_{\text{cyc}} x)^2}{2 \sum_{\text{cyc}} xy + (\lambda - 1) \sum_{\text{cyc}} x^2} \\
 & \Leftrightarrow (\lambda + 1) \left(\sum_{\text{cyc}} x^2 + 2 \sum_{\text{cyc}} xy \right) \stackrel{?}{\geq} 6 \sum_{\text{cyc}} xy + (3\lambda - 3) \sum_{\text{cyc}} x^2 \quad (\because \lambda - 1 \geq 0) \\
 & \Leftrightarrow 2(2 - \lambda) \left(\sum_{\text{cyc}} x^2 - \sum_{\text{cyc}} xy \right) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because 2 \geq \lambda \text{ and } \sum_{\text{cyc}} x^2 \geq \sum_{\text{cyc}} xy \\
 & \therefore \sum_{\text{cyc}} \frac{xy}{x^2 + \lambda y^2} \leq \frac{3}{\lambda + 1} \quad \forall x, y, z > 0 \text{ and } 1 \leq \lambda \leq 2, " = " \text{ iff } x = y = z \text{ (QED)}
 \end{aligned}$$