ROMANIAN MATHEMATICAL MAGAZINE

SP.638. If $x, y, z \geq 1$ and $\lambda \geq 0$ then:

$$\frac{x^3+x+\lambda}{y+z+\lambda}+\frac{y^3+y+\lambda}{z+x+\lambda}+\frac{z^3+z+\lambda}{x+y+\lambda} \geq 3$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

We have:

$$LHS=\sum \frac{x^3+x+\lambda}{y+z+\lambda} \overset{AM-GM}{\geq} 3\sqrt[3]{\prod \frac{x^3+x+\lambda}{y+z+\lambda}} \overset{(1)}{\geq} 3=RHS,$$

where (1) $\Leftrightarrow \sqrt[3]{\prod \frac{x^3+x+\lambda}{y+z+\lambda}} \geq 1 \Leftrightarrow \prod \frac{x^3+x+\lambda}{y+z+\lambda} \geq 1$, which follows from:

$$\frac{x^3+x+\lambda}{x+y+\lambda}\cdot\frac{y^3+y+\lambda}{x+y+\lambda} \geq 1 \Leftrightarrow (x^3+x+\lambda)(y^3+y+\lambda) \geq (x+y+\lambda)^2 \Leftrightarrow$$

$\Leftrightarrow \lambda(x^3+y^3)+xy(x^2+y^2)+x^3y^3 \geq \lambda(x+y)+(x^2+y^2)+xy$, which follows from:

$\lambda(x^3+y^3) \geq \lambda(x+y), xy(x^2+y^2) \geq (x^2+y^2)$ and $x^3y^3 \geq xy$, true for $x, y \geq 1$ and $\lambda \geq 0$.

From $\frac{x^3+x+\lambda}{x+y+\lambda}\cdot\frac{y^3+y+\lambda}{x+y+\lambda} \geq 1$ and the analogs, by multiplying member by member, we obtain:

$\prod \frac{x^3+x+\lambda}{y+z+\lambda} \geq 1$, with equality for $x=y=z=1$.

Equality holds if and only if $x=y=z=1$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{x^3+x+\lambda}{y+z+\lambda} \overset{AM-GM}{\geq} 2.\sum_{cyc} \frac{x^2}{y+z+\lambda}+\lambda.\sum_{cyc} \frac{1}{y+z+\lambda} \overset{Bergstrom}{\geq}$$

$$\frac{2\left(\sum_{cyc} x\right)^2}{2\sum_{cyc} x+3\lambda}+\frac{9\lambda}{2\sum_{cyc} x+3\lambda} \ (\because \lambda \geq 0) \overset{?}{\geq} 3 \Leftrightarrow 2\left(\sum_{cyc} x\right)^2+9\lambda \overset{?}{\geq} 6\sum_{cyc} x$$

$$\Leftrightarrow \left(\sum_{cyc} x\right)\left(\sum_{cyc} x-3\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because x,y,z \geq 1 \Rightarrow \sum_{cyc} x \geq 3 \text{ and so,}$$

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$$\frac{x^3+x+\lambda}{y+z+\lambda}+\frac{y^3+y+\lambda}{z+x+\lambda}+\frac{z^3+z+\lambda}{x+y+\lambda}\geq 3\ \forall\ x,y,z\geq 1\ and\ \lambda\geq 0,$$

$$''=''\ \text{iff}\ x=y=z=1\ (\text{QED})$$

Solution 3 by Jenish Rijal-Bardiya-Nepal

$$\text{Let } S:=x+y+z\geq 3\ \ (\text{as } x,y,z\geq 1).$$

$$\sum_{cyc}\frac{x^3+x+\lambda}{y+z+\lambda}\overset{\text{Bergström}}{\geq}\frac{\left(\sum\sqrt{x^3+x+\lambda}\right)^2}{2(x+y+z)+3\lambda}\overset{\text{AM-GM}}{\geq}\frac{\left(\sum\sqrt{2x^2+\lambda}\right)^2}{2S+3\lambda}$$

$$=\frac{\left(\sum\sqrt{\left(\sqrt{2}x\right)^2+\sqrt{\lambda}^2}\right)^2}{2S+3\lambda}\geq\overset{\text{Minkowski}}{\geq}\frac{\left(\sqrt{\left[\sqrt{2}(x+y+z)\right]^2+\left(3\sqrt{\lambda}\right)^2}\right)^2}{2S+3\lambda}=$$

$$=\frac{2S^2+9\lambda}{2S+3\lambda}=\frac{2S\cdot S+9\lambda}{2S+3\lambda}\overset{S\geq 3}{\geq}\frac{2S\cdot 3+9\lambda}{2S+3\lambda}=3$$

$$Equality\ holds\ if\ and\ only\ if\ \ x=y=z=1.$$