ROMANIAN MATHEMATICAL MAGAZINE

SP.637 If $a,b,c>0, a+b+c=3$ and $\lambda \geq 0$ then:

$$\frac{a+\lambda}{b^2+1}+\frac{b+\lambda}{c^2+1}+\frac{c+\lambda}{a^2+1}\geq \frac{3(\lambda+1)}{2}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

Lemma 1: If $a,b,c>0, a+b+c=3$ then:

$$\frac{a}{b^2+1}+\frac{b}{c^2+1}+\frac{c}{a^2+1}\geq \frac{3}{2}.$$

Proof: We have:

$$\sum \frac{a}{b^2+1}=\sum\left(1-\frac{ab^2}{b^2+1}\right)\overset{AGM}{\geq}\sum\left(a-\frac{ab^2}{2b}\right)=\sum\left(a-\frac{ab}{2}\right)=\sum a-\frac{1}{2}\sum ab=$$

$$=3-\frac{1}{2}\cdot\frac{(\sum a)^2}{3}=3-\frac{1}{2}\cdot\frac{3^2}{3}=\frac{3}{2}.$$

Equality holds if and only if $a=b=c=1$.

Lemma 2: If $a,b,c>0, a+b+c=3$ then:

$$\frac{1}{b^2+1}+\frac{1}{c^2+1}+\frac{1}{a^2+1}\geq \frac{3}{2}.$$

Proof: We have:

$$\sum\frac{1}{b^2+1}=\sum\left(1-\frac{b^2}{b^2+1}\right)\overset{AGM}{\geq}\sum\left(1-\frac{b^2}{2b}\right)=\sum\left(1-\frac{b}{2}\right)=3-\frac{1}{2}\sum a=3-\frac{1}{2}\cdot 3=\frac{3}{2}.$$

Equality holds if and only if $a=b=c=1$. Let's get back to the main problem.

Using the Lemmas, we obtain:

$$LHS=\sum\frac{a+\lambda}{b^2+1}=\sum\frac{a}{b^2+1}+\lambda\sum\frac{1}{b^2+1}\overset{Lemma\ 1+Lemma\ 2}{\geq}$$

$$\geq\frac{3}{2}+\lambda\frac{3}{2}=\frac{3(\lambda+1)}{2}=RHS$$

Equality holds if and only if $a=b=c=1$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a+\lambda}{b^2+1}=\sum_{cyc}\frac{(a+\lambda)(b^2+1-b^2)}{b^2+1}=\sum_{cyc}(a+\lambda)-\sum_{cyc}\frac{b^2(a+\lambda)}{b^2+1}$$

ROMANIAN MATHEMATICAL MAGAZINE

$$\overset{AM-GM}{\geq} \sum_{cyc} a + 3\lambda - \sum_{cyc} \frac{b^2(a+\lambda)}{2b} = 3(\lambda+1) - \frac{1}{2}\sum_{cyc} ab - \frac{\lambda}{2}\sum_{cyc} a \left(\because \sum_{cyc} a = 3 \right)$$

$$\geq 3(\lambda+1) - \frac{1}{6}\left(\sum_{cyc} a\right)^2 - \frac{\lambda}{2}\sum_{cyc} a = 3(\lambda+1) - \frac{9}{6} - \frac{3\lambda}{2} \left(\because \sum_{cyc} a = 3 \right) =$$

$$= \frac{3(\lambda+1)}{2} \text{ and so}$$

$$\frac{a+\lambda}{b^2+1} + \frac{b+\lambda}{c^2+1} + \frac{c+\lambda}{a^2+1} \geq \frac{3(\lambda+1)}{2}$$

$$\forall a, b, c > 0 \,\big|\, a+b+c = 3 \text{ and } \lambda \geq 0, '' = '' \text{ iff } a = b = c = 1 \text{ (QED)}$$