ROMANIAN MATHEMATICAL MAGAZINE

SP.636 Find all positive real numbers such that $ab+bc+ca=3abc$, and

$$\sqrt{\frac{a+b}{a^2+b^2}}+\sqrt{\frac{b+c}{b^2+c^2}}+\sqrt{\frac{c+a}{c^2+a^2}}=3$$

Proposed by José Luis Díaz – Barrero – Spain

Solution 1 by proposer

The condition $ab+bc+ca=3abc$ is equivalent to

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3.$$

Since $2(a^2+b^2)\geq(a+b)^2$ and, by AM-HM, $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 4$, we have

$$\frac{a+b}{a^2+b^2}\leq\frac{2}{a+b}\leq\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right),$$

with equality in both if and only if $a=b$. Similarly, for the other two terms. By the Cauchy-Schwarz inequality,

$$\sum_{cyc}\sqrt{\frac{a+b}{a^2+b^2}}\leq\sqrt{3\sum_{cyc}\frac{a+b}{a^2+b^2}}\leq\sqrt{3\sum_{cyc}\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)}$$

$$=\sqrt{3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}=\sqrt{9}=3.$$

Hence the given equation holds if and only if equality occurs throughout. This requires $a=b, b=c$ and $c=a$. Then $\frac{3}{a}=3$, that is, $a=b=c=1$.

Conversely, $a=b=c=1$ clearly satisfies both conditions. So, the only solution is $(a,b,c)=(1,1,1)$.

Solution 2 by Marin Chirciu-Romania

$$ab+bc+ca=3abc\Leftrightarrow\sum\frac{1}{a}=3.$$

$$LHS=\sum\sqrt{\frac{a+b}{a^2+b^2}}\overset{AM-GM}{\leq}\sum\sqrt{\frac{a+b}{2ab}}\overset{CBS}{\leq}\sqrt{3\sum\frac{a+b}{2ab}}=\sqrt{\frac{3}{2}\sum\left(\frac{1}{a}+\frac{1}{b}\right)}=$$

$$= \sqrt{\frac{3}{2} \cdot 2 \sum \frac{1}{a}} = \sqrt{3 \cdot 3} = 3 = RHS.$$

Equality holds if and only if $a = b = c = 1$.

We deduce that $(a, b, c) = (1, 1, 1)$ is the unique solution of system of conditions.

Remark.

Let be $\lambda \geq 0$ fixed. Find $a, b, c > 0$ such that $ab + bc + ca = 3abc$ and

$$\sum \sqrt{\frac{a + \lambda b}{a^2 + b^2}} = 3\sqrt{\frac{\lambda + 1}{2}}$$

Marin Chirciu

Solution

$$ab + bc + ca = 3abc \Leftrightarrow \sum \frac{1}{a} = 3.$$

$$LHS = \sum \sqrt{\frac{a + \lambda b}{a^2 + b^2}} \overset{AM-GM}{\leq} \sum \sqrt{\frac{a + \lambda b}{2ab}} \overset{CBS}{\leq} \sqrt{3 \sum \frac{a + \lambda b}{2ab}} = \sqrt{\frac{3}{2} \sum \left(\lambda \frac{1}{a} + \frac{1}{b}\right)} =$$

$$= \sqrt{\frac{3}{2} \cdot (\lambda + 1) \sum \frac{1}{a}} = \sqrt{\frac{3}{2} \cdot (\lambda + 1) \cdot 3} = 3\sqrt{\frac{\lambda + 1}{2}} = RHS.$$

Equality holds if and only if $a = b = c = 1$.

We deduce that $(a, b, c) = (1, 1, 1)$ is the unique solution of the system of conditions.