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SP.635 Let $n \geq 2$, and let $a_1, a_2, \ldots, a_n$ be positive real numbers whose sum is $S = \sum_{i=1}^{n} a_i$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be positive real numbers satisfying $\sum_{i=1}^{n} \lambda_i = 1$. Prove that:

$$\sum_{i=1}^{n} \lambda_i \frac{a_i}{S - a_i} \geq \frac{S}{\prod_{i=1}^{n} (S - a_i)^{\lambda_i}} - 1.$$

Determine when equality holds.

Proposed by Mihály Bencze – Romania

Solution 1 by proposer

Since all the a_i are positive, we have $S - a_i > 0$ for every $i \in \{1, 2, \ldots, n\}$. Moreover

$$\frac{a_i}{S - a_i} = \frac{S - (S - a_i)}{S - a_i} = \frac{S}{S - a_i} - 1.$$

Multiplying this identity by λ_i, summing over all i, and using $\sum_{i=1}^{n} \lambda_i = 1$, we obtain

$$\sum_{i=1}^{n} \lambda_i \frac{a_i}{S - a_i} = s \sum_{i=1}^{n} \frac{\lambda_i}{S - a_i} - \sum_{i=1}^{n} \lambda_i = S \sum_{i=1}^{n} \frac{\lambda_i}{S - a_i} - 1.$$

By the weighed arithmetic-geometric mean inequality applied to the positive numbers $\frac{1}{S - a_i}$ with weights λ_i, we have

$$\sum_{i=1}^{n} \frac{\lambda_i}{S - a_i} \geq \prod_{i=1}^{n} \left(\frac{1}{S - a_i} \right)^{\lambda_i} = \frac{1}{\prod_{i=1}^{n} (S - a_i)^{\lambda_i}}.$$

Therefore,

$$\sum_{i=1}^{n} \lambda_i \frac{a_i}{S - a_i} = S \sum_{i=1}^{n} \frac{\lambda_i}{S - a_i} - 1 \geq \frac{S}{\prod_{i=1}^{n} (S - a_i)^{\lambda_i}} - 1,$$

as required. Since every weight λ_i is positive, equality in the weighted arithmetic – geometric mean inequality holds if and only if

$$\frac{1}{S - a_i} = \frac{1}{S - a_2} = \cdots = \frac{1}{S - a_n}$$

This is equivalent to $a_1 = a_2 = \cdots = a_n$. Conversely, if $a_1 = a_2 = \cdots = a_n$, then all the numbers $\frac{1}{S-a_i}$ are equal, so equality indeed holds.

Thus equality occurs if and only if $a_1 = a_2 = \cdots a_n$.

Solution 2 by Jenish Rijal-Bardiya-Nepal

$$\sum_{i=1}^{n} \lambda_i \cdot \frac{a_i}{S-a_i} = \sum_{i=1}^{n} \lambda_i \cdot \left(\frac{S}{S-a_i} - 1\right) = \sum_{i=1}^{n} \lambda_i \cdot \left(\frac{S}{S-a_i}\right) - \sum_{i=1}^{n} \lambda_i = S\sum_{i=1}^{n} \left(\frac{\lambda_i}{S-a_i}\right) - 1$$

Now, $$\sum_{i=1}^{n} \left(\frac{\lambda_i}{S-a_i}\right) \overset{\text{Weighted AM-GM}}{\geq} \prod_{i=1}^{n} \left(\frac{1}{S-a_i}\right)^{\lambda_i} = \frac{1}{\prod_{i=1}^{n}(S-a_i)^{\lambda_i}}$$

Therefore,

$$\sum_{i=1}^{n} \lambda_i \cdot \frac{a_i}{S-a_i} = S\sum_{i=1}^{n} \left(\frac{\lambda_i}{S-a_i}\right) - 1 \geq \frac{S}{\prod_{i=1}^{n}(S-a_i)^{\lambda_i}} - 1$$

Equality holds if and only if $a_1 = a_2 = a_3 = \cdots\cdots = a_n$