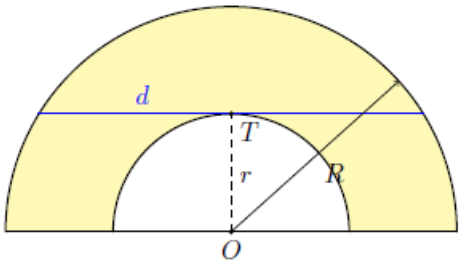
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SP.634 Let two concentric spheres have radii R and r, where $R > r > 0$, and let their common centre be O. A plane parallel to their common equatorial plane is tangent to the smaller sphere in the upper half-space. Its intersection with the larger sphere is a circle of diameter d. The two upper solid hemispheres bound a hemispherical shell. Determine the volume V of this shell in terms of the general data r and d. Also determine the difference S of the curved surface areas of the two hemispheres.



The figure is a meridian section of the three-dimensional configuration.

Proposed by Mihály Bencze – Romania

Solution 1 by proposer

Choose Cartesian coordinates with origin O and with the equatorial plane given by $z = 0$. The upper halves of the two spheres then satisfy

$$x^2 + y^2 + z^2 = R^2, \quad x^2 + y^2 + z^2 = r^2, \quad z \geq 0.$$

The tangent plane in the problem is $z = r$. Indeed, its distance from O is r, so it meets the smaller sphere only at the point $T = (0, 0, r)$. Substituting $z = r$ into the equation of the larger sphere gives

$$x^2 + y^2 = R^2 - r^2.$$

Thus the section of the larger sphere is a circle of radius $\frac{d}{2}$, and hence

$$\left(\frac{d}{2}\right)^2 = R^2 - r^2.$$

Consequently,

$$R = \sqrt{r^2 + \frac{d^2}{4}}. \qquad (1)$$

For completeness, we derive the hemisphere volume. At height $z \in [0, R]$, the horizontal section of the upper ball of radius R is a disk of area $\pi(R^2 - z^2)$. Therefore

$$Vol(H_r) = \int_0^R \pi(R^2 - z^2)\, dz = \pi\left[R^2 z - \frac{z^3}{3}\right]_0^R = \frac{2\pi R^3}{3}$$

The same calculation with radius r gives $Vol(H_r) = \frac{2\pi r^3}{3}$. The required shell is $\frac{H_R}{H_r}$, so it volume is

$$V = \frac{2\pi}{3}(R^3 - r^3).$$

Using (1), we obtain the answer entirely in terms of the given data:

$$V = \frac{2\pi}{3}\left[\left(r^2 + \frac{d^2}{4}\right)^{\frac{3}{2}} - r^3\right].$$

It remains to find the curved-area difference. The curved surface area of a hemisphere of radius a is $2\pi a^2$. Hence

$$S = 2\pi R^2 - 2\pi r^2 = 2\pi(R^2 - r^2) = 2\pi\left(\frac{d}{2}\right)^2,$$

and therefore

$$S = \frac{\pi d^2}{2}.$$

Notice that, unlike the planar area in the original configuration and the curved-area difference above, the volume is not determined by d alone. The formula contains r essentially: different inner radii with the same section diameter d produce shell volumes. This is why both r and d are included among the general data.

Solution 2 by Hikmat Mammadov-Azerbaijan

$$R, r \in \mathbb{R}^+, R > r, d > 0$$

$$R^2 = r^2 + \left(\frac{d}{2}\right)^2 \Rightarrow R^2 - r^2 = \frac{d^2}{4} \Rightarrow R = \left(r^2 + \frac{d^2}{4}\right)^{\frac{1}{2}}$$

$$V = \int_0^r \pi[(R^2 - z^2) - (r^2 - z^2)]\, dz + \int_r^R \pi(R^2 - z^2)\, dz$$

$$\Rightarrow V = \pi\int_0^r (R^2 - z^2 - r^2 + z^2)\, dz + \pi\int_r^R (R^2 - z^2)\, dz$$

$$\Rightarrow V=\pi\int_0^r(R^2-r^2)dz+\pi\int_r^R(R^2-z^2)\,dz$$

$$\Rightarrow V=\pi(R^2-r^2)[z]_0^r+\pi\left[R^2z-\frac{z^3}{3}\right]_r^R$$

$$\Rightarrow V=\pi(R^2-r^2)(r-0)+\pi\left[\left(R^3-\frac{R^3}{3}\right)-\left(R^2r-\frac{r^3}{3}\right)\right]$$

$$\Rightarrow V=\pi(R^2-r^2)r+\pi\left[\frac{2R^3}{3}-R^2r+\frac{r^3}{3}\right]$$

$$\Rightarrow V=\pi R^2r-\pi r^3+\frac{2}{3}\pi R^3-\pi R^2r+\frac{1}{3}\pi r^3$$

$$\Rightarrow V=(\pi R^2r-\pi R^2r)+\left(\frac{1}{3}\pi r^3-\pi r^3\right)+\frac{2}{3}\pi R^3\Rightarrow V=0-\frac{2}{3}\pi r^3+\frac{2}{3}\pi R^3$$

$$\Rightarrow V=\frac{2}{3}\pi R^3-\frac{2}{3}\pi r^3\Rightarrow V=\frac{2}{3}\pi(R^3-r^3)\Rightarrow V=\frac{2}{3}\pi\left((R^2)^{\frac{3}{2}}-r^3\right)$$

$$\Rightarrow V=\frac{2}{3}\pi\left[\left(r^2+\frac{d^2}{4}\right)^{\frac{3}{2}}-r^3\right]\Rightarrow V=\frac{2}{3}\pi\left[\left(\frac{4r^2+d^2}{4}\right)^{\frac{3}{2}}-r^3\right]$$

$$\Rightarrow V=\frac{2}{3}\pi\left[\frac{(4r^2+d^2)^{\frac{3}{2}}}{4^{\frac{3}{2}}}-r^3\right]\Rightarrow V=\frac{2}{3}\pi\left[\frac{(4r^2+d^2)^{\frac{3}{2}}}{8}-r^3\right]$$

$$\Rightarrow V=\frac{2}{3}\pi\left[\frac{(4r^2+d^2)^{\frac{3}{2}}-8r^3}{8}\right]\Rightarrow V=\frac{2}{24}\pi\left[(4r^2+d^2)^{\frac{3}{2}}-8r^3\right]$$

$$\Rightarrow V=\frac{\pi}{12}\left[(4r^2+d^2)^{\frac{3}{2}}-8r^3\right]S=S_R-S_r\Rightarrow S=2\pi R^2-2\pi r^2$$

$$\Rightarrow S=2\pi(R^2-r^2)\Rightarrow S=2\pi\left(\frac{d^2}{4}\right)\Rightarrow S=\frac{2\pi d^2}{4}\Rightarrow S=\frac{\pi d^2}{2}$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$R,r\in\mathbb{R}^+,R>r,d>0$$

$$R^2=r^2+\left(\frac{d}{2}\right)^2\Rightarrow R^2=r^2+\frac{d^2}{4}\Rightarrow R^2-r^2=\frac{d^2}{4}\Rightarrow R=\sqrt{r^2+\frac{d^2}{4}}\Rightarrow R=\left(r^2+\frac{d^2}{4}\right)^{\frac{1}{2}}$$

$$V=V_R-V_r\Rightarrow V=\frac{2}{3}\pi R^3-\frac{2}{3}\pi r^3\Rightarrow V=\frac{2}{3}\pi(R^3-r^3)\Rightarrow V=\frac{2}{3}\pi\left((R^2)^{\frac{3}{2}}-r^3\right)$$

$$\Rightarrow V=\frac{2}{3}\pi\left[\left(r^2+\frac{d^2}{4}\right)^{\frac{3}{2}}-r^3\right]\Rightarrow V=\frac{2}{3}\pi\left[\left(\frac{4r^2+d^2}{4}\right)^{\frac{3}{2}}-r^3\right]$$

$$\Rightarrow V=\frac{2}{3}\pi\left[\frac{(4r^2+d^2)^{\frac{3}{2}}}{4^{\frac{3}{2}}}-r^3\right]\Rightarrow V=\frac{2}{3}\pi\left[\frac{(4r^2+d^2)^{\frac{3}{2}}}{8}-r^3\right]$$

$$\Rightarrow V=\frac{2}{3}\pi\left[\frac{(4r^2+d^2)^{\frac{3}{2}}-8r^3}{8}\right]\Rightarrow V=\frac{2}{24}\pi\left[(4r^2+d^3)^{\frac{3}{2}}-8r^3\right]$$

$$\Rightarrow V=\frac{\pi}{12}\left[(4r^2+d^2)^{\frac{3}{2}}-8r^3\right]$$

$$S=S_R-S_r\Rightarrow S=2\pi R^2-2\pi r^2\Rightarrow S=2\pi(R^2-r^2)\Rightarrow S=2\pi\left(\frac{d^2}{4}\right)$$

$$\Rightarrow S=\frac{2\pi d^2}{4}\Rightarrow S=\frac{\pi d^2}{2}$$