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SP.633 Let $n \geq 3$ be an integer, and let $a_1, a_2, \ldots, a_n$ be positive real numbers such that

$$\sum_{r=1}^{n} a_r^2 = n.$$

For every triple $1 \leq i < j < k \leq n$, put

$$R_{ijk} = \frac{a_i}{3a_i + a_j} + \frac{a_j}{3a_j + a_k} + \frac{a_k}{3a_k + a_i}$$

and

$$D_{ijk} = \sqrt{\frac{\sqrt{3\left(a_i^2 + a_j^2 + a_k^2\right)}}{a_i + a_j + a_k}}.$$

Prove that

$$\frac{1}{\binom{n}{3}} \sum_{1 \leq i < j < k \leq n} \left(\frac{2}{3} R_{ijk} + \frac{3}{2} D_{ijk}\right) \geq 2.$$

Determine all cases of equality.

Proposed by Mihály Bencze – Romania

Solution 1 by proposer

First prove the homogeneous three-variable inequality on which the generalization is based.

Lemma. For all positive real numbers x, y, z, if

$$Q = x^2 + y^2 + z^2, \quad S = x + y + z,$$

then

$$\frac{2}{3}\left(\frac{x}{3x+y} + \frac{y}{3y+z} + \frac{z}{3z+x}\right) + \frac{3}{2}\sqrt{\frac{\sqrt{3Q}}{S}} \geq 2. \qquad (1)$$

Equality holds if and only if $x = y = z$.

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Proof. Both terms on the left-hand side of (1) are unchanged when x, y, z are multiplied by the same positive number. We may therefore assume that

$$Q = x^2 + y^2 + z^2 = 3. \qquad (2)$$

By the Cauchy-Schwarz inequality in Engel form,

$$\frac{x}{3x+y} + \frac{y}{3y+z} + \frac{z}{3z+x} = \frac{x^2}{3x^2+xy} + \frac{y^2}{3y^2+yz} + \frac{z^2}{3z^2+zx}$$
$$\geq \frac{(x+y+z)^2}{3(x^2+y^2+z^2)+xy+yz+zx}.$$

Under (2),

$$xy + yz + zx = \frac{S^2 - Q}{2} = \frac{S^2 - 3}{2}.$$

Consequently,

$$\frac{x}{3x+y} + \frac{y}{3y+z} + \frac{z}{3z+x} \geq \frac{2S^2}{S^2+15} \qquad (3)$$

Moreover, Cauchy – Schwarz gives $S \leq 3$. Set

$$t = \sqrt{\frac{3}{S}};$$

then $t \geq 1$ and $S = \frac{3}{t^2}$. From (3), the left-hand side of (1) is at least

$$\frac{4S^2}{3(S^2+15)} + \frac{3t}{2} = \frac{4}{3+5t^2} + \frac{3t}{2}.$$

It remains to verify that this last expression is at least 2. Indeed,

$$\frac{4}{3+5t^4} + \frac{3t}{2} - 2 = \frac{15t^5 - 20t^4 + 9t - 4}{2(3+5t^4)}$$
$$= \frac{(t-1)(15t^4 - 5t^3 - 5t^2 - 5t + 4)}{2(3+5t^4)}.$$

For $t > 1$,

$$15t^4 - 5t^3 - 5t^2 - 5t + 4 > 15t^4 - 15t^3 + 4$$
$$15t^3(t-1) + 4 > 0.$$

Thus (1) follows,

Equality requires $t = 1$, hence $S = 3$. Together with $Q = 3$, equality in Cauchy – Schwarz yields $x = y = z = 1$ under normalization (2). Removing the normalization, equality holds exactly when $x = y = z$. We now apply the lemma to each triple (a_i, a_j, a_k) with $1 \le i < j < k \le n$. For such a triple, the lemma states exactly that

$$\frac{2}{3}R_{ijk} + \frac{3}{2}D_{ijk} \ge 2 \qquad (4)$$

There are $\binom{n}{3}$ triples. Summing (4) over all of them and dividing by $\binom{n}{3}$ proves the boxed inequality. Finally, equality in the average is possible only if equality holds for every triple. By the equality statement in the lemma, this means that

$$a_i = a_j = a_k \quad \text{for every} \quad 1 \le i < j < k \le n.$$

Hence $a_1 = a_2 = \cdots a_n$. The condition $\sum_{r=1}^{n} a_r^2 = n$ and positive then give

$$a_1 = a_2 = \cdots = a_n = 1.$$

Conversely, these values clearly give equality, so the equality case is completely determined.

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\sum_{1\le i<j<k\le n}\left(a_i^2 + a_j^2 + a_k^2\right) = \sum_{1\le i<j<k\le n} a_i^2 + \sum_{1\le i<j<k\le n} a_j^2 + \sum_{1\le i<j<k\le n} a_k^2$$

$$= \binom{n-1}{2}a_1^2 + \binom{n-1}{2}a_2^2 + \cdots + \binom{n-1}{2}a_n^2 = \binom{n-1}{2}\sum_{r=1}^{n} a_r^2 = \binom{n-1}{2}n$$

$$\Rightarrow \frac{1}{\binom{n}{3}}\sum_{1\le i<j<k\le n}\left(a_i^2 + a_j^2 + a_k^2\right) = \frac{\binom{n-1}{2}n}{\frac{n(n-1)(n-2)}{6}} = \frac{\frac{(n-1)(n-2)}{2}\cdot n}{\frac{n(n-1)(n-2)}{6}} =$$

$$= \frac{(n-1)(n-2)n}{2}\cdot\frac{6}{n(n-1)(n-2)} = 3$$

$$R_{ijk} = \frac{a_i}{3a_i + a_j} + \frac{a_j}{3a_j + a_k} + \frac{a_k}{3a_k + a_i} = \frac{a_i^2}{3a_i^2 + a_i a_j} + \frac{a_j^2}{3a_j^2 + a_j a_k} + \frac{a_k^2}{3a_k^2 + a_k a_i}$$

$$\Rightarrow R_{ijk} \ge \frac{\left(a_i + a_j + a_k\right)^2}{\left(3a_i^2 + a_i a_j\right) + \left(3a_j^2 + aja_k\right) + \left(3a_k^2 + a_k a_i\right)} =$$

$$=\frac{(a_i+a_j+a_k)^2}{3(a_i^2+a_j^2+a_k^2)+(a_ia_j+a_ja_k+a_ka_i)}$$

$$(a_i-a_j)^2+(a_j-a_k)^2+(a_k-a_i)^2\geq 0$$

$$\Rightarrow a_1^2-2a_ia_j+a_j^2+a_j^2-2a_ja_k+a_k^2+a_k^2-2a_ka_i+a_i^2\geq 0$$

$$\Rightarrow 2(a_i^2+a_j^2+a_k^2)-2(a_ia_j+a_ja_k+a_ka_i)\geq 0$$

$$\Rightarrow a_ia_j+a_ja_k+a_ka_i\leq a_i^2+a_j^2+a_k^2\Rightarrow 3(a_i^2+a_j^2+a_k^2)+(a_ia_j+a_ja_k+a_ka_i)\leq$$

$$\leq 3(a_i^2+a_j^2+a_k^2)+(a_i^2+a_j^2+a_k^2)=4(a_i^2+a_j^2+a_k^2)\Rightarrow R_{ijk}\geq\frac{(a_i+a_j+a_k)^2}{4(a_i^2+a_j^2+a_k^2)}$$

$$D_{ijk}=\sqrt{\frac{3(a_i^2+a_j^2+a_k^2)}{a_i+a_j+a_k}}=\frac{\sqrt{3(a_i^2+a_j^2+a_k^2)}}{\sqrt{a_i+a_j+a_k}}$$

$$\sqrt{\frac{a_i^2+a_j^2+a_k^2}{3}}\geq\frac{a_i+a_j+a_k}{3}\Rightarrow a_i+a_j+a_k\leq 3\sqrt{\frac{a_i^2+a_j^2+a_k^2}{3}}=\sqrt{3(a_i^2+a_j^2+a_k^2)}$$

$$\Rightarrow D_{ijk}\geq\frac{\sqrt{3(a_i^2+a_j^2+a_k^2)}}{\sqrt{\sqrt{3(a_i^2+a_j^2+a_k^2)}}}=\sqrt[4]{3(a_i^2+a_j^2+a_k^2)}\geq 1$$

$$\frac{2}{3}R_{ijk}+\frac{3}{2}D_{ijk}\geq 2\sqrt{\left(\frac{2}{3}R_{ijk}\right)\cdot\left(\frac{3}{2}D_{ijk}\right)}=2\sqrt{R_{ijk}\cdot D_{ijk}}$$

$$R_{ijk}\cdot D_{ijk}\geq\frac{(a_i+a_j+a_k)^2}{4(a_i^2+a_j^2+a_k^2)}\cdot\sqrt{\frac{3(a_i^2+a_j^2+a_k^2)}{a_i+a_j+a_k}}=\frac{\sqrt{3}}{4}\cdot\frac{(a_i+a_j+a_k)^{\frac{3}{2}}}{(a_i^2+a_j^2+a_k^2)^{\frac{1}{2}}}$$

$$a_i=a_j=a_k\Rightarrow R_{ijk}=\frac{a_i}{4a_i}+\frac{a_i}{4a_i}+\frac{a_i}{4a_i}=\frac{3}{4},\ \ D_{ijk}=\sqrt{\frac{3(3a_i^2)}{3a_i}}=\sqrt{3a_i}\overset{a_i=1}{=}\sqrt{3}$$

$$\Rightarrow\frac{2}{3}R_{ijk}+\frac{3}{2}D_{ijk}\geq\frac{2}{3}\cdot\frac{3}{4}+\frac{3}{2}\cdot 1=\frac{1}{2}+\frac{3}{2}=2$$

$$\frac{1}{\binom{n}{3}}\sum_{1\leq i<j<k\leq n}\left(\frac{2}{3}R_{ijk}+\frac{3}{2}D_{ijk}\right)\geq\frac{1}{\binom{n}{3}}\sum_{1\leq i<j<k\leq n}2=\frac{2\cdot\binom{n}{3}}{\binom{n}{3}}=2$$

$$\Leftrightarrow a_1 = a_2 = \cdots = a_n \Rightarrow \sum_{r=1}^{n} a_r^2 = n \cdot a_1^2 = n \Rightarrow a_1^2 = 1 \Rightarrow a_1 = a_2 = \cdots = a_n = 1$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\sum_{1\le i<j<k\le n} (a_i^2 + a_j^2 + a_k^2) = \sum_{1\le i<j<k\le n} a_i^2 + \sum_{1\le i<j<k\le n} a_j^2 + \sum_{1\le i<j<k\le n} a_k^2$$

$$= \binom{n-1}{2} a_1^2 + \binom{n-1}{2} a_2^2 + \cdots + \binom{n-1}{2} a_n^2 = \binom{n-1}{2} \sum_{r=1}^{n} a_r^2 = \binom{n-1}{2} n$$

$$\Rightarrow \frac{1}{\binom{n}{3}} \sum_{1\le i<j<k\le n} (a_i^2 + a_j^2 + a_k^2) = \frac{\binom{n-1}{2} n}{\frac{n(n-1)(n-2)}{6}} = \frac{\frac{(n-1)(n-2)}{2} \cdot n}{\frac{n(n-1)(n-2)}{6}} =$$

$$= \frac{(n-1)(n-2)n}{2} \cdot \frac{6}{n(n-1)(n-2)} = 3$$

$$R_{ijk} = \frac{a_i}{3a_i + a_j} + \frac{a_j}{3a_j + a_k} + \frac{a_k}{3a_k + a_i} = \frac{a_i^2}{3a_i^2 + a_i a_j} + \frac{a_j^2}{3a_j^2 + a_j k} + \frac{a_k^2}{3a_k^2 + a_k a_i}$$

$$\Rightarrow R_{ijk} \ge \frac{(a_i + a_j + a_k)^2}{(3a_i^2 + a_i a_j) + (3a_j^2 + a_j a_k) + (3a_k^2 + a_k a_i)} =$$

$$= \frac{(a_i + a_j + a_k)^2}{3(a_i^2 + a_j^2 + a_k^2) + (a_i a_j + a_j a_k + a_k a_i)}$$

$$(a_i - a_j)^2 + (a_j - a_k)^2 + (a_k - a_i)^2 \ge 0 \Rightarrow$$

$$\Rightarrow a_i^2 - 2a_i a_j + a_j^2 + a_j^2 - 2a_j a_k + a_k^2 + a_k^2 - 2a_k a_i + a_i^2 \ge 0$$

$$\Rightarrow 2(a_i^2 + a_j^2 + a_k^2) - 2(a_i a_j + a_j a_k + a_k a_i) \ge 0 \Rightarrow a_i a_j + a_j a_k + a_k a_i \le a_1^2 + a_j^2 + a_k^2$$

$$\Rightarrow 3(a_i^2 + a_j^2 + a_k^2) + (a_i a_j + a_j a_k + a_k a_i) \le$$

$$\le 3(a_i^2 + a_j^2 + a_k^2) + (a_i^2 + a_j^2 + a_k^2) = 4(a_i^2 + a_j^2 + a_k^2) \Rightarrow R_{ijk} \ge \frac{(a_i + a_j + a_k)^2}{4(a_i^2 + a_j^2 + a_k^2)}$$

$$\sqrt{\frac{a_i^2 + a_j^2 + a_k^2}{3}} \ge \frac{a_i + a_j + a_k}{3} \Rightarrow a_i + a_j + a_k \le 3\sqrt{\frac{a_i^2 + a_j^2 + a_k^2}{3}} = \sqrt{3(a_i^2 + a_j^2 + a_k^2)}$$

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$$\Rightarrow D_{ijk}=\sqrt{\frac{3\left(a_i^2+a_j^2+a_k^2\right)}{a_i+a_j+a_k}}\geq\frac{\sqrt{3\left(a_i^2+a_j^2+a_k^2\right)}}{\sqrt{\sqrt{3\left(a_i^2+a_j^2+a_k^2\right)}}}=\sqrt[4]{3\left(a_i^2+a_j^2+a_k^2\right)}\geq 1$$

$$\frac{2}{3}R_{ijk}+\frac{3}{2}D_{ijk}\geq 2\sqrt{\left(\frac{2}{3}R_{ijk}\right)\cdot\left(\frac{3}{2}D_{ijk}\right)}=2\sqrt{R_{ijk}\cdot D_{ijk}}$$

$$R_{ijk}\cdot D_{ijk}\geq\frac{\left(a_i+a_j+a_k\right)^2}{4\left(a_i^2+a_j^2+a_k^2\right)}\cdot\sqrt{\frac{3\left(a_i^2+a_j^2+a_k^2\right)}{a_i+a_j+a_k}}=\frac{\sqrt{3}}{5}\cdot\frac{\left(a_i+a_j+a_k\right)^{\frac{3}{2}}}{\left(a_i^2+a_j^2+a_k^2\right)^{\frac{1}{2}}}$$

$$a_i=a_j=a_k\Rightarrow R_{ijk}=\frac{3}{4},\qquad D_{ijk}=\sqrt{3a_i}\overset{a_i=1}{=}\sqrt{3}\Rightarrow\frac{2}{3}\left(\frac{3}{4}\right)+\frac{3}{2}(1)=\frac{1}{2}+\frac{3}{2}=2$$

$$\frac{1}{\binom{n}{3}}\sum_{1\leq i<j<k\leq n}\left(\frac{2}{3}R_{ijk}+\frac{3}{2}D_{ijk}\right)\geq\frac{1}{\binom{n}{3}}\sum_{1\leq i<j<k\leq n}2=\frac{2\cdot\binom{n}{3}}{\binom{n}{3}}=2$$

$$\Leftrightarrow a_1=a_2=\cdots=a_n\Rightarrow\sum_{r=1}^{n}a_r^2=n\cdot a_1^2=n\Rightarrow a_1^2=1\Rightarrow a_1=a_2=\cdots=a_n=1$$

