ROMANIAN MATHEMATICAL MAGAZINE

SP.632 Prove that 2 is the largest positive value of the constant k such that

$$a^2+b^2+c^2+d^2-4\geq k(ab+bc+cd+da-4)$$

for all positive real numbers a,b,c,d with at most one of them larger than 1 and $abcd=1$.

Proposed by Vasile Cîrtoaje – Romania

Solution 1 by proposer

For $b=d=1$ and $ac=1$, the constraints are satisfied and the inequality becomes as follows:

$$a^2+c^2-4\geq 2k(a+c-2),\ \ a^2+\frac{1}{a^2}-4\geq 2k\left(a+\frac{1}{a}-2\right),$$

$$\frac{(a^2-1)^2}{a^2}\geq\frac{2k(a-1)^2}{a},\ \ (a-1)^2f(a)\geq 0,$$

where $f(a)=(a+1)^2-2ka$. The necessary condition $f(1)\geq 0$ implies $k\leq 2$. To show that 2 is the largest positive value of k, we need to prove that

$$a^2+b^2+c^2+d^2-2(ab+bc+cd+da)+4\geq 0.$$

According to Lemma below, it suffices to show that $E\geq 0$ for $a\geq 1\geq b\geq c\geq d$, where

$$E=a^2+b^2+c^2+d^2-2(a+d)(b+c)+4.$$

We have

$$E=(a+d-b-c)^2-2(ad+bc-2).$$

Let $x=bc\leq 1$. From $(a-1)(d-1)\leq 0$, we get

$$a+d\geq ad+1=\frac{1}{x}+1,$$

and from $(b-1)(c-1)\geq 0$, we get $b+c\leq bc+1=x+1$. Since

$$a+d-b-c\geq\frac{1}{x}+1-(x+1)=\frac{1-x^2}{x}\geq 0,$$

we have

$$E\geq\frac{(1-x^2)^2}{x^2}-2\left(\frac{1}{x}+x-2\right)=\frac{(1-x^2)^2}{x^2}-\frac{2(1-x)^2}{x}=\frac{(1-x)^2(1+x^2)}{x^2}\geq 0.$$

The proof is completed. If $k=2$, then the equality occurs for $a=b=c=d=1$.

Lemma. Let x_1, x_2, x_3, x_4 be real numbers, and let (a, b, c, d) be a permutation of the sequence (x_1, x_2, x_3, x_4) such that $a \geq b \geq c \geq d$. Then,

$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 \leq (a + d)(b + c).$$

Proof. It suffices to show that

$$(a + d)(b + c) \geq (a + c)(b + d) \geq (a + b)(c + d).$$

The left inequality reduces to $(a - b)(c - d) \geq 0$,

and the right inequality to $(a - d)(b - c) \geq 0$.

Solution 2 by Jenish Rijal-Bardiya-Nepal

Let $(a, b, c, d) = \left(t, 1, \frac{1}{t}, 1\right)$; $t > 1$

Notice: $\frac{a^2 + b^2 + c^2 + d^2 - 4}{ab + bc + cd + da - 4} = \frac{t^2 + \frac{1}{t^2} - 2}{2\left(t + \frac{1}{t} - 2\right)} = 2 + \frac{(t-1)^2}{2t} \longrightarrow 2^+ \quad (\text{as } t \longrightarrow 1^+)$

Hence no $k > 2$ can work. It remains to prove that $k_{\max} = 2$ works.

If all $a, b, c, d \leq 1$, then $a = b = c = d = 1$ and the inequality becomes trivial.

Otherwise, due to symmetry, assume that: $a > 1$ and $b, c, d \leq 1$.

Here, $ac = \frac{1}{bd} \geq 1 \geq bd \Leftrightarrow ac \geq bd$. Now,

$(a - 1)(1 - c) \geq 0 \Leftrightarrow a + c \geq ac + 1$ and $(1 - b)(1 - d) \geq 0$

$\Leftrightarrow b + d \leq bd + 1$ Thus, $a + c - b - d \geq ac - bd \geq 0$

Now,

$$(a^2 + b^2 + c^2 + d^2 - 4) - 2(ab + bc + cd + da - 4) = (a + c - b - d)^2 - 2(ac + bd - 2) \geq$$

$$\overset{a+c-b-d \geq ac-bd}{\geqslant} (ac - bd)^2 - 2(ac + bd - 2) \overset{abcd=1}{=} (ac + bd)(ac + bd - 2) \overset{?}{\geqslant} 0$$

, which is true because $ac + bd \overset{AM-GM}{\geqslant} 2\sqrt{abcd} = 2$.

Therefore: $a^2 + b^2 + c^2 + d^2 - 4 \geq 2(ab + bc + cd + da - 4)$

Hence: $k_{\max} = 2$. *Equality holds for* $a = b = c = d = 1$.