ROMANIAN MATHEMATICAL MAGAZINE

SP.631 Let a, b, c be nonnegative real numbers such that at most one of them is larger than 1 and $ab + bc + ca = 3$. Prove that:

$$a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}} \geq 3$$

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Solution 1 by proposer

Assume that $a \geq 1 \geq b \geq c$, and write the inequality as $F \geq 0$, where

$$F = a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}} - 3.$$

For fixed c, assume that a and F are functions of b. By differentiating the equality constraint, we get $(b + c)a' + a + c = 0$.

Therefore, we have

$$\frac{3F'(b)}{2} = b^{-\frac{1}{3}} + a^{-\frac{1}{3}}a' = b^{-\frac{1}{3}} - \frac{a^{-\frac{1}{3}}(a + c)}{b + c}.$$

We will show that $F'(b) \leq 0$, which is equivalent to

$$\frac{a + c}{b + c} \geq \left(\frac{a}{b}\right)^{\frac{1}{3}}.$$

This is true if

$$\left(\frac{a + c}{b + c}\right)^3 \geq \frac{a}{b}.$$

Since

$$\left(\frac{a + c}{b + c}\right)^3 \geq \left(\frac{a + c}{b + c}\right)^2 \geq \frac{2(a + c)}{b + c} - 1,$$

it suffices to show that

$$\frac{2(a + c)}{b + c} - 1 \geq \frac{a}{b},$$

which is equivalent to

$$\frac{2(a + c)}{b + c} - 2 \geq \frac{a}{b} - 1, \qquad \frac{2(a - b)}{b + c} \geq \frac{a - b}{b}, \qquad \frac{(a - b)(b - c)}{b(b + c)} \geq 0.$$

From $F'(b) \leq 0$, it follows that $F(b)$ is decreasing and has the minimum value when b is maximum, hence when $b = 1$. So, it suffices to consider the case $b = 1$, when we need to show that $a^{\frac{2}{3}} + c^{\frac{2}{3}} \geq 2$ for $ac + a + c = 3$. Replacing a and c with a^3 and c^3, respectively, we need to prove that $a^2 + c^2 \geq 2$ for $a^3c^3 + a^3 + c^3 = 3$. By contradiction method, we will show that $a^2 + c^2 < 2$ involves $a^3c^3 + a^3 + c^3 < 3$. Since the expression $a^3c^3 + a^3 + c^3$ is strictly increasing in each of its variables a, b, c, it is enough to prove that $a^2 + c^2 = 2$ involves $a^3c^3 + a^3 + c^3 \leq 3$. Denoting $x = \frac{a+c}{2}$, we have $ac = 2x^2 - 1$. From $2x^2 - 1 = ac \geq 0$ and $2x^2 - 1 = ac \leq x^2$, we get $\frac{1}{\sqrt{2}} \leq x \leq 1$, hence $\frac{2}{3} < x \leq 1$. Write the required inequality as follows:

$$a^3c^3 + (a + c)^3 - 3ac(a + c) \leq 3, \quad (2x^2 - 1)^3 + 8x^3 - 6x(2x^2 - 1) \leq 3,$$
$$4x^6 - 6x^4 - 2x^3 + 3x^2 + 3x - 2 \leq 0, \quad (x - 1)f(x) \leq 0,$$

where

$$f(x) = 4x^5 + 4x^4 - 2x^3 - 4x^2 - x + 2.$$

We need to show that $f(x) \geq 0$. Indeed, we have

$$f(x) > 3x^5 + 4x^4 - 2x^3 - 4x^2 - x + 2$$
$$= (3x-2)(x-1)^2(x^2+4x+1) + 20x^3 - 22x^2 + 4$$
$$\geq 20x^3 - 22x^2 + 4 = \frac{80x^3 - 88x^2 + 16}{4} = \frac{(4x-3)^2(5x+2) + 3x - 2}{4} > 0.$$

The proof is completed. The equality occurs for $a = b = c = 1$.

Solution 2 by Soumava Chakraborty-Kolkata-India

When one 1 variable equals to zero and WLOG we may assume $a = 0$, then $bc = 3\ (b, c > 0)$ and b, c cannot be both ≤ 1 (since then : $3 = bc \leq 1$)

$\therefore$ WLOG we may assume $b > 1 \geq c > 0$; then : $b^{\frac{2}{3}} + c^{\frac{2}{3}} = c^{\frac{2}{3}} + \left(\frac{3}{c}\right)^{\frac{2}{3}} = f(c)$ (say)

and then : $f'(c) = \frac{2}{\alpha}\left(\frac{1}{3} - \frac{1}{\sqrt[3]{3}.\alpha^4}\right)\ (\alpha = \sqrt[3]{c} \leq 1) \overset{\alpha \leq 1}{\leq} \frac{2}{\alpha}\left(\frac{1}{3} - \frac{1}{\sqrt[3]{3}.}\right) < 0 \Rightarrow$

$f(c)$ is $\downarrow$ on $(0, 1] \Rightarrow f(c) \geq f(1) > 3$ and we now focus on the case when : $a, b, c > 0$ and if $a, b, c \leq 1$, then : $3 = ab + bc + ca \leq 3 \Rightarrow a = b = c = 1$

$\Rightarrow a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}} = 3$ and now, since at most one of $a, b, c > 1$, then : WLOG we may assume $x^2 > 1 \Rightarrow y^2, z^2 \leq 1\ (\sqrt[3]{a} = x, \sqrt[3]{b} = y, \sqrt[3]{c} = z)$

$$\therefore \prod_{cyc}(x^2 - 1) \geq 0 \Rightarrow \sum_{cyc} x^2 \geq 1 + \sum_{cyc} x^2y^2 - x^2y^2z^2 \overset{?}{\geq} 3$$

$$\Leftrightarrow \sqrt[3]{\frac{\sum_{cyc} x^3y^3}{3}}.3\sum_{cyc} x^2y^2 \overset{?}{\geq} 2\sum_{cyc} x^3y^3 + 3x^2y^2z^2 \left(\because \sum_{cyc} x^3y^3 = 3\right)$$

$$\Leftrightarrow 9\left(\sum_{cyc} u^3\right)\left(\sum_{cyc} u^2\right)^3 \overset{?}{\underset{(*)}{\geq}} \left(2\sum_{cyc} u^3 + 3uvw\right)^3 \quad (u = xy, v = yz, w = zx)$$

Assigning $v + w = a', w + x = b', u + v = c' \Rightarrow a' + b' - c' = 2w > 0, b' + c' - a' = 2u > 0$ and $c' + a' - b' = 2v > 0 \Rightarrow a' + b' > c', b' + c' > a', c' + a' > b'$

$\Rightarrow a', b', c'$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $\sum_{cyc} u = s \Rightarrow uvw = r^2s, \sum_{cyc} u^2 = s^2 - 8Rr - 2r^2,$

$\sum_{cyc} u^3 = s^3 - 12Rrs$ and then : $(*) \Leftrightarrow s^8 - (36Rr + 90r^2)s^6 +$

$$r^2(864R^2 + 2376Rr + 54r^2)s^4 - r^3(11520R^3 + 19008R^2r + 1512Rr^2 + 99r^3)s^2$$
$$+Rr^4(55296R^3 + 41472R^2r + 10368Rr^2 + 864r^3) \overset{?}{\underset{(**)}{\geq}} 0$$

Case 1 $28R - 110r \geq 0$ and then : $P = (s^2 - 16Rr + 5r^2)^4 +$

$r(28R - 110r)(s^2 - 16Rr + 5r^2)^3 \overset{\text{via successive Gerretsen}}{\geq} 0 \therefore$ in order to prove $(**)$,

it suffices to prove : LHS of $(**) \overset{?}{\geq} P \Leftrightarrow (672R^2 - 2364Rr + 1554r^2)s^4 -$

$r(16640R^3 - 63552R^2r + 51612Rr^2 - 7651r^3)s^2 +$

$$64r^2\left((1632R^4-6792R^3r+6687R^2r^2-1979Rr^3+205r^4)+30Rr^3+5r^4\right)\overset{?}{\geq}(\bullet)\,0\ \text{and}\ \because Q=\overbrace{(672R^2-2364Rr+1554r^2)}^{>0\ \because 28R-110r\geq 0}(s^2-16Rr+5r^2)^2+$$

$$r\overbrace{(4864R^3-18816R^2r+21756Rr^2-7889r^3)}^{>0\ \because 28R-110r\geq 0}(s^2-16Rr+5r^2)\overset{\text{via successive Gerretsen}}{\geq}0$$

$$\therefore \text{in order to prove } (\bullet), \text{it suffices to prove : LHS of } (\bullet)\overset{?}{\geq} Q$$

$$\Leftrightarrow (t-2)\left((t-2)(512t^2-320t+536)+729\right)\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\rightarrow \text{true}\ \because t\overset{\text{Euler}}{\geq}2$$

$$\Rightarrow (\bullet)\Rightarrow (**)\ \text{is true}$$

Case 2 $28R-110r<0$ *and then* : $T=(s^2-16Rr+5r^2)^4+$

$$r(28R-110r)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)+$$

$$r\overbrace{(112R^3-552R^2r+1080Rr^2+124r^3)}^{>0}(s^2-16Rr+5r^2)^2$$

$$\overset{\text{via successive Gerretsen, Double-Rouche and Gerretsen}}{\geq} 0\ \therefore \text{in order to prove } (**),$$

$$\text{it suffices to prove : LHS of } (**)\overset{?}{\geq} T\Leftrightarrow$$

$$\overbrace{(1792R^4-8224R^3r+10656R^2r^2-2252Rr^3-1729r^4)}^{X}s^2\overset{?}{\geq}(\bullet\bullet)\ r\left(\begin{matrix}64(448R^5-2328R^4r+3815R^3r^2-1982R^2r^3-27Rr^4+58r^5)+\\48R^3r^2+24R^2r^3+24Rr^4+13r^5\end{matrix}\right)$$

Case 2(*a*) $X\geq 0$ *and then* : LHS of $(\bullet\bullet)\overset{\text{Gerretsen}}{\geq}X(16Rr-5r^2)\overset{?}{\geq}$ RHS of $(\bullet\bullet)$

$$\Leftrightarrow (t-2)(704t^3-1308t^2+510t-205)\overset{?}{\geq}0\rightarrow \text{true}\ \because t\overset{\text{Euler}}{\geq}2$$

Case 2(b) $X<0$; *then* : $LHS_{(\bullet\bullet)}\overset{\text{Gerretsen}}{\geq}X(4R^2+4Rr+3r^2)\overset{?}{\geq}RHS_{(\bullet\bullet)}$

$$\Leftrightarrow (t-2)\left((t-2)\left((t-2)(1792t^3-2848t^2+2432t+4288)+9477\right)+2916\right)$$

$$\overset{?}{\geq}0\rightarrow \text{true}\ \because t\overset{\text{Euler}}{\geq}2\ \therefore (\bullet\bullet)\Rightarrow (**)\ \text{is true}\ \therefore \text{combining } \textit{all cases},$$

$(**)$ is true $\forall XYZ_{s,R,r}\Rightarrow (*)$ is true & rounding up, $a^{\frac{2}{3}}+b^{\frac{2}{3}}+c^{\frac{2}{3}}\geq 3\ \forall\ a,b,c\geq 0$ such that *at most one of them is larger than 1 and* $ab+bc+ca=3$, $''=''$ iff $a=b=c=1$ (QED)

Solution 3 by Jenish Rijal-Bardiya-Nepal

$$\text{Let } x=a^{\frac{2}{3}},\quad y=b^{\frac{2}{3}},\quad \textit{and}\ \ z=c^{\frac{2}{3}}.\ \text{Then}\ \ (xy)^{\frac{3}{2}}+(yz)^{\frac{3}{2}}+(zx)^{\frac{3}{2}}\overset{(\blacksquare)}{\geq}3.$$

We need to prove $x+y+z\geq 3$. *Suppose, for a contradiction that* $x+y+z<3$.

Clearly if all $x,y,z\leq 1$, *then* $(\blacksquare)$ *forces* $x=y=z=1$.

Thus, due to symmetry, assume that:

$$x>1\geq y,z\geq 0.$$

Now, $2(xy)^{\frac{3}{2}} \overset{AM-GM}{\geq} xy(x+y)$ $(and\ analogs)$

Adding analogous inequalities, we get:

$$2(xy)^{\frac{3}{2}} + 2(yz)^{\frac{3}{2}} + 2(zx)^{\frac{3}{2}} \leq xy(x+y) + yz(y+z) + zx(z+x)$$

$$\Leftrightarrow 6 \leq x^2(y+z) + x(y^2+z^2) + yz(y+z)$$

But, $6 - x^2(y+z) - x(y^2+z^2) - yz(y+z) =$

$$= (3-x-y-z)[(x+1)(y+z)+2] + (1-y)(1-z)(2x-y-z) \text{ which is } > 0.$$

which contradicts with $x^2(y+z) + x(y^2+z^2) + yz(y+z) \geq 6$.

So, our assumption that $x+y+z<3$ *is wrong*.

Therefore,

$$a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}} \geq 3.$$

Equality holds if and only if $a=b=c=1$.