ROMANIAN MATHEMATICAL MAGAZINE

SOLVED RMM-PROBLEMS BY SAMUEL AJIBADE

By Samuel Adekunle Ajibade- Nigeria

01 Prove that:

$$\int_0^{\infty} \mathrm{Li}_3(-\mathrm{e}^{-x})\,\mathrm{sech}(x)\,\mathrm{d}x = \frac{\pi^2 G}{6} - \frac{3\pi\zeta(3)}{64} - 2\beta(4)$$

Where

$\mathrm{Li}_3(z)$ *is the trilogarithm function,*
$G = \beta(2)$ *is Catalan's constant*
$\zeta(3)$ *is Apery's constant,* $\beta(s)$ *is the Dirichlet beta function*

Proposed by: Samuel Adekunle Ajibade- Nigeria

Solution 1 by: Emedi Abner Ngoy- DR Congo

$$\int_0^{\infty} \mathrm{Li}_3\left(-e^{-x}\right)\mathrm{sech}(x)\,dx,\ \text{let } e^{-x} = u, \qquad \mathrm{sech}\,x = \frac{2e^{-x}}{1+e^{-2x}}$$

$$I = 2\int_0^1 \frac{\mathrm{Li}_3(-u)}{1+u^2}\,du = 2\left(\int_0^{\infty} - \int_1^{\infty}\right)\frac{\mathrm{Li}_3(-u)}{1+u^2}\,du$$

$$\int_1^{\infty} \frac{\mathrm{Li}_3(-u)}{1+u^2}\,du,\ \text{let } u \to \frac{1}{u}, \qquad \int_0^1 \frac{\mathrm{Li}_3\left(-\frac{1}{u}\right)}{1+u^2}\,du$$

$$I + I = 2\left(\int_0^{\infty} \frac{\mathrm{Li}_3(-u)}{1+u^2}\,du + \int_0^1 \frac{\mathrm{Li}_3(-u) - \mathrm{Li}_3\left(-\frac{1}{u}\right)}{1+u^2}\,du\right)$$

$$\mathrm{Li}_3(-u) - \mathrm{Li}_3\left(-\frac{1}{u}\right) = -\frac{1}{6}\ln^3 u - \frac{\pi^2}{6}\ln u$$

$$I = \int_0^{\infty} \frac{\mathrm{Li}_3(-u)}{1+u^2}\,du - \frac{1}{6}\int_0^1 \frac{\ln^3 u}{1+u^2}\,du - \frac{\pi^2}{6}\int_0^1 \frac{\ln u}{1+u^2}\,du$$

$$I_1 = \int_0^{\infty} \frac{\mathrm{Li}_3(-u)}{1+u^2}\,du = \int_0^{\infty} \frac{1}{1+u^2}\left(-\frac{1}{2}\int_0^1 \frac{u\ln^2 y}{(1+u^2)(1+uy)}\,dy\right)du$$

$$I_1 = -\frac{1}{2}\int_0^1 \ln^2 y\left(\int_0^{\infty} \frac{u}{(1+u^2)(1+uy)}\,du\right)dy$$

$$I_1 = -\frac{1}{2}\int_0^1 \ln^2 y\left(\frac{\pi y}{2(1+y^2)} - \frac{\ln y}{1+y^2}\right)dy$$

$$I_1=-\frac{\pi}{4}\int_0^1\frac{y\ln^2 y}{1+y^2}\,dy+\frac{1}{2}\int_0^1\frac{\ln^3 y}{1+y^2}\,dy,\ y^2\to y$$

$$I_1=-\frac{\pi}{32}\int_0^1\frac{\ln^2 y}{1+y}\,dy+\frac{1}{2}\int_0^1\frac{\ln^3 y}{1+y^2}\,dy$$

$$I=\frac{1}{3}\int_0^1\frac{\ln^3 u}{1+u^2}\,du-\frac{\pi}{32}\int_0^1\frac{\ln^2 u}{1+u}\,du-\frac{\pi^2}{6}\int_0^1\frac{\ln u}{1+u^2}\,du$$

$$\int_0^1\frac{\ln^{n-1}u}{1+u^2}\,du=\frac{\beta(n)\,\Gamma(n)}{(-1)^{n-1}}$$

$$\int_0^1\frac{\ln^n u}{1+u}\,du=(-1)^n\,n!\,(1-2^{-n})\zeta(n+1)$$

$$I=\frac{1}{3}\left(\frac{\beta(4)\,\Gamma(4)}{(-1)^{4-1}}\right)-\frac{\pi^2}{32}\left(2\,(-1)^2\left(1-\frac{1}{4}\right)\zeta(3)\right)-\frac{\pi^2}{6}\left(\frac{\beta(2)\,\Gamma(2)}{(-1)^{2-1}}\right)$$

$$I=-2\beta(4)-\frac{3\pi}{64}\zeta(3)+\frac{\pi^2}{6}\beta(2)$$

$$I=\frac{\pi^2}{6}G-\frac{3\pi}{64}\zeta(3)-2\beta(4)$$

Solution 2 by: Pham Duc Nam- Vietnam

$$I=\int_0^\infty \mathrm{Li}_3\,(-e^{-x})\mathrm{sech}(x)\mathrm{d}x=\frac{\pi^2}{6}G-\frac{3\pi}{64}\zeta(3)-2\beta(4)$$

1.)

$$a>0\colon\ T(a)=\int_0^\infty\frac{\ln^2(x)}{(x+a)(x^2+1)}\mathrm{d}x$$
$$=\frac{1}{1+a^2}\left(-\frac{1}{3}\ln^3(a)-\frac{\pi^2}{3}\ln(a)+\frac{\pi^3}{8}a\right)$$

Proof :

Some basic integrals :

$$C=\int_0^\infty\frac{\ln^2(x)}{1+x^2}\mathrm{d}x=\frac{\pi^3}{8},\int_0^\infty\frac{\ln^2(x)}{(1+x)^2}\mathrm{d}x=\frac{\pi^2}{3}$$

And :

$$\int_0^\infty\frac{\ln(x)}{(1+x)^2}\mathrm{d}x=0$$

Observation :

$$\frac{1+a^2}{(x+a)(x^2+1)}=\frac{1}{x+a}+\frac{a-x}{1+x^2}$$

$$\Rightarrow (1+a^2)T(a)-aC=\int_0^{\infty}\ln^2(x)\left(\frac{1}{x+a}-\frac{x}{1+x^2}\right)dx$$

But :

$$\int_0^{\infty}\ln^2(x)\left(\frac{1}{x+1}-\frac{x}{1+x^2}\right)dx=0 \text{ by } x\to\frac{1}{x}$$

$$\Rightarrow (1+a^2)T(a)-aC=\int_0^{\infty}\ln^2(x)\left(\frac{1}{x+a}-\frac{1}{x+1}\right)dx$$

$$=\int_0^{\infty}\ln^2(x)\left(\int_a^1\frac{1}{(x+t)^2}dt\right)dx=\int_a^1\left(\int_0^{\infty}\frac{\ln^2(x)}{(x+t)^2}dx\right)dt$$

$$\underset{x=tu}{\smile}$$

$$=\int_a^1\frac{1}{t}\left(\int_0^{\infty}\frac{(\ln(t)+\ln(u))^2}{(1+u)^2}du\right)dt$$

$$=\int_a^1\frac{1}{t}\left(\ln^2(t)+\frac{\pi^2}{3}\right)dt=-\frac{1}{3}\ln^3(a)-\frac{\pi^2}{3}\ln(a)$$

$$\Rightarrow T(a)=\frac{1}{1+a^2}\left(-\frac{1}{3}\ln^3(a)-\frac{\pi^2}{3}\ln(a)+aC\right)$$

$$=\frac{1}{1+a^2}\left(-\frac{1}{3}\ln^3(a)-\frac{\pi^2}{3}\ln(a)+\frac{\pi^3}{8}a\right), \text{ proved.}$$

From here, integrate both sides w.r.t a from 0 to 1 we have :

$$\int_0^{\infty}\frac{\ln^2(x)}{1+x^2}\ln\left(\frac{1+x}{x}\right)dx=\frac{\pi^2}{3}G+\frac{\pi^3}{16}\ln(2)+2\beta(4)$$

$$\text{But : } \int_0^{\infty}\frac{\ln^3(x)}{1+x^2}dx=0\Rightarrow\int_0^{\infty}\frac{\ln^2(x)}{1+x^2}\ln\left(\frac{1+x}{x}\right)dx=\int_0^{\infty}\frac{\ln^2(x)}{1+x^2}\ln(1+x)dx$$

$$=\int_0^1\frac{\ln^2(x)}{1+x^2}\ln(1+x)dx+\int_1^{\infty}\frac{\ln^2(x)}{1+x^2}\ln(1+x)dx$$

$$\underset{x\to\frac{1}{x}}{\smile}$$

$$=2\int_0^1\frac{\ln^2(x)}{1+x^2}\ln(1+x)dx-\int_0^1\frac{\ln^3(x)}{1+x^2}dx$$

$$=2\int_0^1\frac{\ln^2(x)}{1+x^2}\ln(1+x)dx+6\beta(4)$$

$$\Rightarrow \int_0^1 \frac{\ln^2(x)}{1+x^2}\ln(1+x)dx = \frac{\pi^2}{6}G + \frac{\pi^3}{32}\ln(2) - 2\beta(4)$$

2.) $I = \int_0^\infty \mathrm{Li}_3(-e^{-x})\mathrm{sech}(x)dx \overset{t=e^{-x}}{=} 2\int_0^1 \frac{\mathrm{Li}_3(-t)}{1+t^2}dt$

$$= 2\int_0^1 \frac{1}{1+t^2}\left(-\frac{t}{2}\int_0^1 \frac{\ln^2(u)}{1+ut}du\right)dt$$

$$= -\int_0^1 \ln^2(u)\left(\int_0^1 \frac{t}{(1+t^2)(1+ut)}dt\right)du$$

$$= -\int_0^1 \frac{\ln^2(u)}{1+u^2}\left(\frac{1}{2}\ln(2) + \frac{\pi}{4}u - \ln(1+u)\right)du$$

$$* \int_0^1 \frac{\ln^2(u)}{1+u^2}du = 2\beta(3) = \frac{\pi^3}{16}$$

$$* \int_0^1 \frac{u\ln^2(u)}{1+u^2}du = \frac{1}{8}\int_0^1 \frac{\ln^2(u)}{1+u}du = \frac{3}{16}\zeta(3)$$

$$* \int_0^1 \frac{\ln^2(u)}{1+u^2}\ln(1+u)du \overset{\text{From 1.)}}{=} \frac{\pi^2}{6}G + \frac{\pi^3}{32}\ln(2) - 2\beta(4)$$

$$\Rightarrow I = -\frac{1}{2}\ln(2)\left(\frac{\pi^3}{16}\right) - \frac{\pi}{4}\left(\frac{3}{16}\zeta(3)\right) + \frac{\pi^2}{6}G + \frac{\pi^3}{32}\ln(2) - 2\beta(4)$$

$$I = \frac{\pi^2}{6}G - \frac{3\pi}{64}\zeta(3) - 2\beta(4)$$

Solution 3 by: Yang Silva Cartolin-Peru

$$\Omega = \int_0^\infty \mathrm{Li}_3(-e^{-x})\mathrm{sech}(x)dx = \frac{\pi^2}{6}G - \frac{3\pi}{64}\zeta(3) - 2\beta(4)$$

$$\Omega = 2\int_0^\infty \frac{\mathrm{Li}_3(-e^{-x})}{e^x + e^{-x}}dx = 2\int_0^\infty \frac{\mathrm{Li}_3(-e^{-x})}{1+e^{-2x}}dx$$

$$\overset{t=e^{-x}}{\Rightarrow} \Omega = 2\int_0^1 \frac{\mathrm{Li}_3(-t)}{1+t^2}dt, \text{I. B. P:} \begin{cases} u = \mathrm{Li}_3(-t) \to du = \dfrac{\mathrm{Li}_2(-t)}{t}dt \\ dv = \dfrac{dt}{1+t^2} \to v = \arctan(t) \end{cases}$$

$$\Omega = -\frac{3\pi}{8}\zeta(3) - 2\int_0^1 \frac{\arctan(t)\mathrm{Li}_2(-t)}{t}dt, \text{I. B. P:} \begin{cases} u = \mathrm{Li}_2(-t) \to du = \dfrac{\ln(1+t)}{t}dt \\ dv = \dfrac{\arctan(t)}{t}dt \to v = \mathrm{Ti}_2(t) \end{cases}$$

$$\Omega = -\frac{3\pi}{8}\zeta(3) + \frac{\pi^2}{6}G - 2\int_0^1 \frac{\mathrm{Ti}_2(t)\ln(1+t)}{t}dt$$

$$\Omega=-\frac{3\pi}{8}\zeta(3)+\frac{\pi^2}{6}G-2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\int_0^1 t^{2n}\ln(1+t)\,dt$$

■We know that: $\int_0^1 x^{2n}\ln(1+x)\,dx=\frac{2\ln(2)}{2n+1}-\frac{1}{(2n+1)^2}+\frac{H_n}{2n+1}-\frac{H_{2n}}{2n+1}$

$$\Omega=-\frac{3\pi}{8}\zeta(3)+\frac{\pi^2}{6}G-4\ln(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}+2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4}$$
$$-2\sum_{n=0}^{\infty}\frac{(-1)^nH_n}{(2n+1)^3}+2\sum_{n=0}^{\infty}\frac{(-1)^nH_{2n}}{(2n+1)^3}$$

■ We know that:

↪ Dirichlet Beta Function: $\beta(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s},\beta(3)=\frac{\pi^3}{32}$

$$\hookrightarrow\sum_{n=0}^{\infty}\frac{(-1)^nH_n}{(2n+1)^3}=3\beta(4)-\frac{7\pi}{16}\zeta(3)-\frac{\pi^3}{16}\ln(2)$$

$$\hookrightarrow\sum_{n=0}^{\infty}\frac{(-1)^nH_{2n}}{(2n+1)^3}=\beta(4)-\frac{35\pi}{128}\zeta(3)$$

Therefore: $\Omega=-\frac{3\pi}{8}\zeta(3)+\frac{\pi^2}{6}G-\frac{\pi^3}{8}\ln(2)+2\beta(4)$
$$-2\left(3\beta(4)-\frac{7\pi}{16}\zeta(3)-\frac{\pi^3}{16}\ln(2)\right)+2\left(\beta(4)-\frac{35\pi}{128}\zeta(3)\right)$$

$$\Omega=\frac{\pi^2}{6}G-\frac{3\pi}{64}\zeta(3)-2\beta(4)\ \ \therefore$$

02 Evaluate in closed form:

$$\int_0^{\infty}\frac{x^a\ln^2(x)\sinh(bx)}{e^{cx}-1}dx$$

Where

$$a>-1,c>b>0,\{a,b,c\ \in\mathbb{R}\}$$

Proposed by: Samuel Adekunle Ajibade- Nigeria

Solution 1 by: Archit Jain- India

$$\sinh(bx)=\frac{e^{bx}-e^{-bx}}{2}$$

$$I=\int_0^{\infty}\frac{x^a(\ln x)^2\sinh(bx)}{e^{cx}-1}\mathrm{d}x$$

Let

$$I(s)=\int_0^{\infty}\frac{x^{s-1}\sinh(bx)}{e^{cx}-1}\mathrm{d}x$$

Differentiating with respect to s,

$$I'(s)=\int_0^{\infty}\frac{x^{s-1}\ln(x)\sinh(bx)}{e^{cx}-1}\mathrm{d}x,\qquad I''(s)=\int_0^{\infty}\frac{x^{s-1}\ln^2(x)\sinh(bx)}{e^{cx}-1}\mathrm{d}x$$

Our target is to find

$$I''(a+1)=I$$

Now,

$$\frac{1}{e^{cx}-1}=\frac{e^{-cx}}{1-e^{-cx}}$$

For $c>0$ and $x\in(0,\infty)$, we have

$$0<e^{-cx}<1$$

Then e^{-cx} satisfies the infinite geometric progression:

$$\sum_{n=1}^{\infty}(e^{-cx})^n=\sum_{n=1}^{\infty}e^{-ncx}$$

Therefore,

$$I(s)=\int_0^{\infty}x^{s-1}\left(\frac{e^{bx}-e^{-bx}}{2}\right)\left(\sum_{n=1}^{\infty}e^{-ncx}\right)\mathrm{d}x$$

$$I(s)=\frac{1}{2}\int_0^{\infty}\sum_{n=1}^{\infty}x^{s-1}\left(e^{bx}e^{-ncx}-e^{-bx}e^{-ncx}\right)\mathrm{d}x$$

$$=\frac{1}{2}\int_0^{\infty}\sum_{n=1}^{\infty}x^{s-1}\left(e^{-(nc-b)x}-e^{-(nc+b)x}\right)\mathrm{d}x$$

Now x^{s-1}, $e^{-(nc-b)x}$ and $e^{-(nc+b)x}$ are all positive for $x>0$, assuming $s>0$.

As given $c>b>0$, we have $nc-b>0$ for all $n\geq 1$.

We apply Tonelli's theorem to interchange the order of sum and integral.

$$I(s)=\frac{1}{2}\sum_{n=1}^{\infty}\int_0^{\infty}\left[x^{s-1}e^{-(nc-b)x}-x^{s-1}e^{-(nc+b)x}\right]\mathrm{d}x$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left[\int_0^{\infty}x^{s-1}\,e^{-(nc-b)x}\mathrm{d}x-\int_0^{\infty}x^{s-1}\,e^{-(nc+b)x}\mathrm{d}x\right]$$

Now,

$$\int_0^{\infty}u^{s-1}\,e^{-u}\mathrm{d}u=\Gamma(s)$$

Let

$$J(\alpha)=\int_0^{\infty}x^{s-1}\,e^{-\alpha x}\mathrm{d}x,\quad \alpha>0$$

Put

$$u=\alpha x,\quad \mathrm{d}x=\frac{\mathrm{d}u}{\alpha}$$

$$J(\alpha)=\int_0^{\infty}\left(\frac{u}{\alpha}\right)^{s-1}e^{-u}\frac{\mathrm{d}u}{\alpha}=\frac{1}{\alpha^s}\int_0^{\infty}u^{s-1}\,e^{-u}\mathrm{d}u=\frac{\Gamma(s)}{\alpha^s}$$

$I(s)$ becomes

$$I(s)=\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{\Gamma(s)}{(nc-b)^s}-\frac{\Gamma(s)}{(nc+b)^s}\right]$$

$$I(s)=\frac{\Gamma(s)}{2}\sum_{n=1}^{\infty}\left[\frac{1}{(nc-b)^s}-\frac{1}{(nc+b)^s}\right]$$

Now we know the Hurwitz zeta function:

$$\zeta(s,q)=\sum_{k=0}^{\infty}\frac{1}{(k+q)^s}$$

$$I(s)=\frac{\Gamma(s)}{2c^s}\sum_{n=1}^{\infty}\left[\frac{1}{(n-b/c)^s}-\frac{1}{(n+b/c)^s}\right]$$

Let

$$k=n-1\Rightarrow n=k+1$$

When $n=1, k=0$. Substitute $n=k+1$ in summation:

$$n-\frac{b}{c}=k+1-\frac{b}{c}\quad\text{and}\quad n+\frac{b}{c}=k+1+\frac{b}{c}$$

$$I(s)=\frac{\Gamma(s)}{2c^s}\sum_{k=0}^{\infty}\left[\frac{1}{(k+1-b/c)^s}-\frac{1}{(k+1+b/c)^s}\right]$$

$$I(s)=\frac{\Gamma(s)}{2c^s}\left[\zeta\left(s,1-\frac{b}{c}\right)-\zeta\left(s,1+\frac{b}{c}\right)\right]$$

Now,

$$I(s)=F(s)G(s)$$

where

$$F(s)=\frac{\Gamma(s)}{2c^s}$$

Taking logarithms,

$$\ln F(s)=\ln\Gamma(s)-\ln 2-s\ln c$$

Using

$$\frac{\Gamma'(s)}{\Gamma(s)}=\psi(s)$$

we get

$$F'(s)=F(s)[\psi(s)-\ln c]$$

Also,

$$G(s)=\zeta\left(s,1-\frac{b}{c}\right)-\zeta\left(s,1+\frac{b}{c}\right)$$

Therefore,

$$G'(s)=\zeta^{(1,0)}\left(s,1-\frac{b}{c}\right)-\zeta^{(1,0)}\left(s,1+\frac{b}{c}\right)$$

Now,

$$F''(s)=F'(s)[\psi(s)-\ln c]+F(s)\psi^{(1)}(s)=F(s)\{[\psi(s)-\ln c]^2+\psi^{(1)}(s)\}$$

Also,

$$G''(s)=\zeta^{(2,0)}\left(s,1-\frac{b}{c}\right)-\zeta^{(2,0)}\left(s,1+\frac{b}{c}\right)$$

Using the product rule,

$$I''(s)=F''(s)G(s)+2F'(s)G'(s)+F(s)G''(s)$$

$$=F(s)\{[\psi(s)-\ln c]^2+\psi^{(1)}(s)\}G(s)$$

$$+2F(s)[\psi(s)-\ln c]G'(s)+F(s)G''(s)$$

$$= F(s)\left\{\left([\psi(s)-\ln c]^2+\psi^{(1)}(s)\right)G(s)+2[\psi(s)-\ln c]G'(s)+G''(s)\right\}$$

Now put $s=a+1$. We know that

$$F(a+1)=\frac{\Gamma(a+1)}{2c^{a+1}}$$

And $I=I''(a+1)$

Hence,

$$I=\frac{\Gamma(a+1)}{2c^{a+1}}\left(\left([\psi(a+1)-\ln c]^2+\psi^{(1)}(a+1)\right)\left[\zeta\left(a+1,1-\frac{b}{c}\right)-\zeta\left(a+1,1+\frac{b}{c}\right)\right]+2[\psi(a+1)-\ln c]\left[\zeta^{(1,0)}\left(a+1,1-\frac{b}{c}\right)-\zeta^{(1,0)}\left(a+1,1+\frac{b}{c}\right)\right]+\left[\zeta^{(2,0)}\left(a+1,1-\frac{b}{c}\right)-\zeta^{(2,0)}\left(a+1,1+\frac{b}{c}\right)\right]\right)$$

Solution 2 by: Yang Silva Cartolin-Peru

$$I=\frac{1}{2}\int_0^\infty \frac{x^a\ln^2(x)}{e^{cx}-1}\left(e^{bx}-e^{-bx}\right)dx$$

$$=\frac{1}{2}\left(\int_0^\infty \frac{x^a e^{bx}\ln^2(x)}{e^{cx}-1}\,dx-\int_0^\infty \frac{x^a e^{-bx}\ln^2(x)}{e^{cx}-1}\,dx\right)$$

$$I=\frac{1}{2}\left(\int_0^\infty \frac{x^a e^{(b-c)x}\ln^2(x)}{1-e^{-cx}}\,dx-\int_0^\infty \frac{x^a e^{-(b+c)x}\ln^2(x)}{1-e^{-cx}}\,dx\right)$$

$$I=\frac{1}{2}\sum_{n=0}^{\infty}\left(\int_0^\infty x^a\,e^{-(-b+c+cn)x}\ln^2(x)\,dx-\int_0^\infty x^a\,e^{-(b+c+cn)x}\ln^2(x)\,dx\right)$$

$$\hookrightarrow \text{Let: } \xi_k=\int_0^\infty x^{s-1}\,e^{-kx}\ln^2(x)\,dx$$

$$\overset{t=kx\to\frac{dt}{k}=dx}{\Rightarrow}\quad \xi_k=\frac{1}{k^s}\int_0^\infty t^{s-1}\,e^{-t}\ln^2\left(\frac{t}{k}\right)dt=\frac{1}{k^s}\int_0^\infty t^{s-1}\,e^{-t}\big(\ln(t)-\ln(k)\big)^2\,dt$$

$$\xi_k=\frac{1}{k^s}\left(\int_0^\infty t^{s-1}\,e^{-t}\ln^2(t)\,dt-2\ln(k)\int_0^\infty t^{s-1}\,e^{-t}\ln(t)\,dt+\ln^2(k)\int_0^\infty t^{s-1}\,e^{-t}\,dt\right)$$

$$\text{We know that: } \Gamma(z)=\int_0^\infty x^{z-1}\,e^{-x}\,dx$$

$$\rightarrow \Gamma'(z)=\int_0^{\infty} x^{z-1}\ln(x)e^{-x}\,dx, \Gamma''(z)=\int_0^{\infty} x^{z-1}\ln^2(x)e^{-x}\,dx$$

$$\Rightarrow \xi_k=\frac{1}{k^s}\left(\Gamma''(s)-2\ln(k)\Gamma'(s)+\ln^2(k)\Gamma(s)\right)\ \therefore$$

Therefore:

$$I=\frac{\Gamma''(a+1)}{2}\sum_{n=0}^{\infty}\left(\frac{1}{(cn+c-b)^{a+1}}-\frac{1}{(cn+c+b)^{a+1}}\right)$$
$$-\Gamma'(a+1)\sum_{n=0}^{\infty}\left(\frac{\ln(cn+c-b)}{(cn+c-b)^{a+1}}-\frac{\ln(cn+c+b)}{(cn+c+b)^{a+1}}\right)$$
$$+\frac{\Gamma(a+1)}{2}\sum_{n=0}^{\infty}\left(\frac{\ln^2(cn+c-b)}{(cn+c-b)^{a+1}}-\frac{\ln^2(cn+c+b)}{(cn+c+b)^{a+1}}\right)$$

$$I=\frac{\Gamma''(a+1)}{2}S_1-\Gamma'(a+1)S_2-\frac{\Gamma(a+1)}{2}S_3$$

$$\hookrightarrow S_1=\sum_{n=0}^{\infty}\left(\frac{1}{(cn+c-b)^{a+1}}-\frac{1}{(cn+c+b)^{a+1}}\right)$$

$$S_1=\frac{1}{c^{a+1}}\sum_{n=0}^{\infty}\left(\frac{1}{\left(n+1-\frac{b}{c}\right)^{a+1}}-\frac{1}{\left(n+1+\frac{b}{c}\right)^{a+1}}\right)$$

$$\textbf{Hurwitz Zeta Function: }\zeta(s,q)=\sum_{k=0}^{\infty}\frac{1}{(k+q)^s}$$

$$\Rightarrow S_1=\frac{1}{c^{a+1}}\left(\zeta\left(a+1,1-\frac{b}{c}\right)-\zeta\left(a+1,1+\frac{b}{c}\right)\right)\ \therefore$$

$$\hookrightarrow S_2=\sum_{n=0}^{\infty}\left(\frac{\ln(cn+c-b)}{(cn+c-b)^{a+1}}-\frac{\ln(cn+c+b)}{(cn+c+b)^{a+1}}\right)$$

$$S_2=\frac{1}{c^{a+1}}\sum_{n=0}^{\infty}\left(\frac{\ln(c)+\ln\left(n+1-\frac{b}{c}\right)}{\left(n+1-\frac{b}{c}\right)^{a+1}}-\frac{\ln(c)+\ln\left(n+1+\frac{b}{c}\right)}{\left(n+1+\frac{b}{c}\right)^{a+1}}\right)$$

$$S_2 = \frac{\ln(c)}{c^{a+1}}\left(\zeta\left(a+1,1-\frac{b}{c}\right)-\zeta\left(a+1,1+\frac{b}{c}\right)\right)$$
$$+\frac{1}{c^{a+1}}\sum_{n=0}^{\infty}\left(\frac{\ln\left(n+1-\frac{b}{c}\right)}{\left(n+1-\frac{b}{c}\right)^{a+1}}-\frac{\ln\left(n+1+\frac{b}{c}\right)}{\left(n+1+\frac{b}{c}\right)^{a+1}}\right)$$

$$S_2 = \frac{\ln(c)}{c^{a+1}}\left(\zeta\left(a+1,1-\frac{b}{c}\right)-\zeta\left(a+1,1+\frac{b}{c}\right)\right)$$
$$-\frac{1}{c^{a+1}}\left(\zeta^{(1,0)}\left(a+1,1-\frac{b}{c}\right)-\zeta^{(1,0)}\left(a+1,1+\frac{b}{c}\right)\right)\ \therefore$$

$$\hookrightarrow S_3 = \sum_{n=0}^{\infty}\left(\frac{\ln^2(cn+c-b)}{(cn+c-b)^{a+1}}-\frac{\ln^2(cn+c+b)}{(cn+c+b)^{a+1}}\right)$$

$$S_3 = \frac{1}{c^{a+1}}\sum_{n=0}^{\infty}\left(\frac{\left(\ln(c)+\ln\left(n+1-\frac{b}{c}\right)\right)^2}{\left(n+1-\frac{b}{c}\right)^{a+1}}-\frac{\left(\ln(c)+\ln\left(n+1+\frac{b}{c}\right)\right)^2}{\left(n+1+\frac{b}{c}\right)^{a+1}}\right)$$

$$S_3 = \frac{\ln^2(c)}{c^{a+1}}\left(\zeta\left(a+1,1-\frac{b}{c}\right)-\zeta\left(a+1,1+\frac{b}{c}\right)\right)$$
$$-\frac{2\ln(c)}{c^{a+1}}\left(\zeta^{(1,0)}\left(a+1,1-\frac{b}{c}\right)-\zeta^{(1,0)}\left(a+1,1+\frac{b}{c}\right)\right)$$
$$+\frac{1}{c^{a+1}}\sum_{n=0}^{\infty}\left(\frac{\ln^2\left(n+1-\frac{b}{c}\right)}{\left(n+1-\frac{b}{c}\right)^{a+1}}-\frac{\ln^2\left(n+1+\frac{b}{c}\right)}{\left(n+1+\frac{b}{c}\right)^{a+1}}\right)$$

$$S_3 = \frac{\ln^2(c)}{c^{a+1}}\left(\zeta\left(a+1,1-\frac{b}{c}\right)-\zeta\left(a+1,1+\frac{b}{c}\right)\right)$$
$$-\frac{2\ln(c)}{c^{a+1}}\left(\zeta^{(1,0)}\left(a+1,1-\frac{b}{c}\right)-\zeta^{(1,0)}\left(a+1,1+\frac{b}{c}\right)\right)$$
$$+\frac{1}{c^{a+1}}\left(\zeta^{(2,0)}\left(a+1,1-\frac{b}{c}\right)-\zeta^{(2,0)}\left(a+1,1+\frac{b}{c}\right)\right)\ \therefore$$

Thus:

$$I=\frac{\Gamma(a+1)}{2c^{a+1}}\Bigg(\Big([\psi(a+1)-\ln c]^2+\psi^{(1)}(a+1)\Big)\Big[\zeta\Big(a+1,1-\frac{b}{c}\Big)-\zeta\Big(a+1,1+\frac{b}{c}\Big)\Big]+2[\psi(a+1)-\ln c]\Big[\zeta^{(1,0)}\Big(a+1,1-\frac{b}{c}\Big)-\zeta^{(1,0)}\Big(a+1,1+\frac{b}{c}\Big)\Big]+\Big[\zeta^{(2,0)}\Big(a+1,1-\frac{b}{c}\Big)-\zeta^{(2,0)}\Big(a+1,1+\frac{b}{c}\Big)\Big]\Bigg)$$

03 Find:

$$I(\alpha)=\int_0^{\frac{\pi}{2}}\ln(\tan(x))\tan(x)\left(\frac{\pi^2}{4}\sin^{\alpha}(x)-x^2\right)dx$$

$$\forall\,\alpha>-2$$

Proposed by: Samuel Adekunle Ajibade- Nigeria

Solution 1 by: Pham Duc Nam- Vietnam

$$\alpha>-2:\ I(\alpha)=\int_0^{\frac{\pi}{2}}\ln\big(\tan(x)\big)\tan(x)\left(\frac{\pi^2}{4}\sin^{\alpha}(x)-x^2\right)\mathrm{d}x$$

1.) Reducing the given integral

Note that :

$$I(\alpha)=I(0)+\frac{\pi^2}{4}D(\alpha)$$

Where :

$$D(\alpha)=\int_0^{\frac{\pi}{2}}\ln\big(\tan(x)\big)\tan(x)(\sin^{\alpha}(x)-1)\mathrm{d}x$$

And letting : $t=\sin(x)$ we have :

$$D(\alpha)=\frac{1}{4}\int_0^1\frac{t^{\frac{\alpha}{2}}-1}{1-t}\ln\left(\frac{t}{1-t}\right)\mathrm{d}t$$

Set : $s=\frac{\alpha}{2}>-1\Rightarrow$

$$4D(\alpha)=\int_0^1\frac{t^s-1}{1-t}\ln(t)\mathrm{d}t-\int_0^1\frac{t^s-1}{1-t}\ln(1-t)\mathrm{d}t$$

$$*\because\ \int_0^1\frac{t^s-1}{1-t}\mathrm{d}t=-H_s=-\psi(1+s)-\gamma,\int_0^1\frac{\ln(t)}{1-t}\mathrm{d}t=-\frac{\pi^2}{6}$$

$$\Rightarrow \int_0^1 \frac{t^s-1}{1-t}\ln(t)\mathrm{d}t = -\frac{\partial}{\partial s}(\psi(1+s)+\gamma) - \int_0^1 \frac{\ln(t)}{1-t}\mathrm{d}t = \frac{\pi^2}{6} - \psi^{(1)}(1+s)$$

$$* \int_0^1 \frac{t^s-1}{1-t}\ln(1-t)\mathrm{d}t = \lim_{u\to -1^+}\frac{\partial}{\partial u}\int_0^1 (t^s-1)(1-t)^u\mathrm{d}t$$

$$= \lim_{u\to -1^+}\frac{\partial}{\partial u}\left(\mathrm{B}(s+1,u+1) - \frac{1}{1+u}\right)$$

$$= \frac{1}{2}\left(\left(\gamma+\psi(1+s)\right)^2 + \frac{\pi^2}{6} - \psi^{(1)}(1+s)\right) \Rightarrow I(\alpha) = I(0) + \frac{\pi^2}{4}D(\alpha)$$

$$= I(0) + \frac{\pi^2}{32}\left(\frac{\pi^2}{6} - \psi^{(1)}\left(1+\frac{\alpha}{2}\right) - \left(\gamma+\psi\left(1+\frac{\alpha}{2}\right)\right)^2\right)$$

The rest of this solution is evaluating of $I(0)$

2.) Evaluating of $I(0)$

$*$Let : $\alpha = 2$ we have :

$$I(2) = \int_0^{\frac{\pi}{2}} \ln\left(\tan(x)\right)\tan(x)\left(\frac{\pi^2}{4}\sin^2(x) - x^2\right)\mathrm{d}x$$

Then the difference of :

$$I(2) - I(0) = -\int_0^{\frac{\pi}{2}} \ln\left(\tan(x)\right)\sin(x)\cos(x)\mathrm{d}x$$

And by substitution : $x \to \frac{\pi}{2} - x$ we immediately have : $I(2) - I(0) = 0$

So, instead of calculating $I(0)$, we can calculate $I(2)$

Set :

$$p = \frac{\pi}{2}, q(x) = p^2\sin^2(x) - x^2 \Rightarrow I(0) = I(2) = \int_0^p q(x)\ln\left(\tan(x)\right)\tan(x)\mathrm{d}x$$

Since : $q(0) = q(p) = 0$, consider the sine Fourier series :

$$q(x) = \sum_{n=1}^{\infty} b_n \sin(2nx)$$

Where :

$$b_n = \frac{4}{\pi}\int_0^p q(x)\sin(2nx)\mathrm{d}x$$

Integration by parts gives us :

$$\int_0^p x^2\sin(2nx)\mathrm{d}x = -\frac{p^2}{2}\frac{(-1)^n}{n} + \frac{1}{4}\frac{(-1)^n-1}{n^3}$$

And :

$$\sin^2(x)=\frac{1}{2}\big(1-\cos(2x)\big)\Rightarrow\int_0^p \sin^2(x)\sin(2nx)\mathrm{d}x$$
$$=\begin{cases}\frac{1}{2n}, & \text{if } n \text{ is odd}\\ -\frac{1}{2}\frac{n}{n^2-1}, & \text{if } n \text{ is even}\end{cases}$$

Combine two formulas, we have :

$$b_n=\begin{cases}\frac{2}{\pi}\frac{1}{n^3}, & \text{if } n \text{ is odd}\\ -\frac{\pi}{2}\frac{1}{n(n^2-1)}, & \text{if } n \text{ is even}\end{cases}$$

Now we define :

$$S_n=\int_0^p \sin(2nx)\ln\big(\tan(x)\big)\tan(x)\mathrm{d}x,$$

using :

$$\ln\big(\tan(x)\big)=-2\sum_{k\geq 1,\ k \text{is odd}}\frac{\cos(2kx)}{k}$$

But :

$$\sin(2nx)\tan(x)=(-1)^{n+1}+2\sum_{k=1}^{n-1}(-1)^{n+k+1}\cos(2kx)-\cos(2nx)$$

Then by orthogonality of cosine :

$$\int_0^p \ln\big(\tan(x)\big)\cos(2kx)\mathrm{d}x=\begin{cases}-\frac{\pi}{2k}, & \text{if } k \text{ is odd}\\ 0, \text{if } k \text{ is even}\end{cases}$$

Now set :

$$h_n=\sum_{1\leq k<n,\ k \text{ is odd}}\frac{1}{k}\Rightarrow S_n=\begin{cases}\pi h_n+\frac{\pi}{2n}, & \text{if } n \text{ is odd}\\ -\pi h_n, \text{if } n \text{ is even}\end{cases}$$

Replace to $I(0)$, we have :

$$I(0)=\sum_{n=1}^{\infty}b_n S_n=2\sum_{n\geq 1,\ n \text{ is odd}}\frac{h_n}{n^3}+\sum_{n\geq 1,\ n \text{ is odd}}\frac{1}{n^4}+\frac{\pi^2}{2}\sum_{n\geq 2,\ n \text{ is even}}\frac{h_n}{n(n^2-1)}$$

$$*\ \text{Let: } Q=\sum_{n\geq 1,\ n \text{ is odd}}\frac{h_n}{n^3}, E=\sum_{n\geq 2,\ n \text{ is even}}\frac{h_n}{n(n^2-1)}, \sum_{n\geq 1,\ n\text{is odd}}\frac{1}{n^4}$$
$$=\Big(1-\frac{1}{16}\Big)\sum_{n\geq 1}\frac{1}{n^4}=\frac{15}{16}\zeta(4)=\frac{\pi^4}{96}$$

Using generating function :

$$\sum_{n\geq 2,\, n \text{ is even}} h_n\, t^n = \frac{t\operatorname{arctanh}(t)}{1-t^2}$$

and

$$\frac{1}{n(n^2-1)} = \frac{1}{2}\int_0^1 t^{n-2}\,(1-t)^2 \mathrm{d}t$$

$$\Rightarrow E = \frac{1}{2}\int_0^1 \frac{1-t}{t(1+t)}\operatorname{arctanh}(t)\mathrm{d}t = \frac{1}{2}\int_0^1 \frac{\operatorname{arctanh}(t)}{t}\mathrm{d}t - \int_0^1 \frac{\operatorname{arctanh}(t)}{1+t}\mathrm{d}t$$

By substitution $u = \frac{1-t}{1+t}$ we easily find : $\int_0^1 \frac{\operatorname{arctanh}(t)}{t}\mathrm{d}t = \frac{\pi^2}{8}, \int_0^1 \frac{\operatorname{arctanh}(t)}{1+t}\mathrm{d}t = \frac{\pi^2}{24}$

$$\Rightarrow E = \frac{1}{2}\frac{\pi^2}{8} - \frac{\pi^2}{24} = \frac{\pi^2}{48} \Rightarrow I(0) = 2Q + \frac{\pi^4}{96} + \frac{\pi^2}{2}E = \frac{\pi^4}{48} + 2Q$$

Using generating function :

$$\sum_{n\geq 1,\, n \text{ is odd}} h_n\, t^n = \frac{t^2\operatorname{arctanh}(t)}{1-t^2}$$

and

$$\frac{1}{n^3} = \frac{1}{2}\int_0^1 t^{n-1}\ln^2(t)\mathrm{d}t$$

$$\Rightarrow Q = \frac{1}{2}\int_0^1 \frac{t\ln^2(t)}{1-t^2}\operatorname{arctanh}(t)\mathrm{d}t = -\frac{1}{4}\int_0^1 \ln^2(t)\operatorname{arctanh}(t)\mathrm{d}\left(\ln(1-t^2)\right)$$

$$\overset{\text{IBPs}}{=} \frac{1}{4}\int_0^1 \frac{\ln^2(t)\ln(1-t^2)}{1-t^2}\mathrm{d}t + \frac{1}{4}\int_0^1 \frac{\ln(t)}{t}\ln(1-t^2)\operatorname{arctanh}(t)\mathrm{d}t$$

Since :

$$\ln(1-t^2)\operatorname{arctanh}(t) = \frac{1}{2}\left(\ln^2(1+t) - \ln^2(1-t)\right)$$

$$\Rightarrow Q = \frac{1}{4}(A + B_+ - B_-) \text{where} : A = \int_0^1 \frac{\ln^2(t)\ln(1-t^2)}{1-t^2}\mathrm{d}t,$$

$$B_\pm = \int_0^1 \frac{\ln(t)}{t}\ln^2(1\pm t)\mathrm{d}t$$

$$*\, A = \int_0^1 \frac{\ln^2(t)\ln(1-t^2)}{1-t^2}\mathrm{d}t \overset{u=t^2}{=} \frac{1}{8}\int_0^1 u^{-\frac{1}{2}}\ln^2(u)\frac{\ln(1-u)}{1-u}\mathrm{d}u$$

From section 1.) :

$$s > -1: \int_0^1 \frac{t^s-1}{1-t}\ln(1-t)\mathrm{d}t = \frac{1}{2}\left(\left(\gamma + \psi(1+s)\right)^2 + \frac{\pi^2}{6} - \psi^{(1)}(1+s)\right)$$

$$\Rightarrow A=\frac{1}{8}\frac{\partial^2}{\partial s^2}\Big|_{s=-\frac{1}{2}}\int_0^1\frac{t^s-1}{1-t}\ln(1-t)\mathrm{d}t=\frac{1}{8}\left(28\ln(2)\zeta(3)-\frac{\pi^4}{4}\right)$$

$$=-\frac{\pi^4}{32}+\frac{7}{2}\ln(2)\zeta(3)$$

$$*\ B_-=\int_0^1\frac{\ln(t)}{t}\ln^2(1-t)\mathrm{d}t$$

From section 1.) :

$$s>-1:\int_0^1\frac{t^s-1}{1-t}\ln(1-t)\mathrm{d}t=\frac{1}{2}\left(\left(\gamma+\psi(1+s)\right)^2+\frac{\pi^2}{6}-\psi^{(1)}(1+s)\right)$$

$$\text{But}:\int_0^1\frac{t^s-1}{1-t}\ln(1-t)\mathrm{d}t\overset{\text{IBPs}}{=}\underbrace{\left[-\frac{1}{2}\ln^2(1-t)(t^s-1)\right]_0^1}_{=0}$$

$$+\frac{s}{2}\int_0^1 t^{s-1}\ln^2(1-t)\mathrm{d}t$$

$$\Rightarrow\int_0^1 t^{s-1}\ln^2(1-t)\mathrm{d}t=\frac{1}{s}\left(\left(\gamma+\psi(1+s)\right)^2+\frac{\pi^2}{6}-\psi^{(1)}(1+s)\right)$$

$$\Rightarrow B_-=\lim_{s\to0^+}\frac{\partial}{\partial s}\left(\frac{1}{s}\left(\left(\gamma+\psi(1+s)\right)^2+\frac{\pi^2}{6}-\psi^{(1)}(1+s)\right)\right)=-\frac{1}{2}\zeta(4)=-\frac{\pi^4}{180}$$

$$*\ B_+=\int_0^1\frac{\ln(x)}{x}\ln^2(1+x)\mathrm{d}x,\text{let}:t=\frac{1}{1+x}\Rightarrow B_+$$

$$=\int_{\frac{1}{2}}^1\frac{\ln^2(t)}{t(1-t)}\big(\ln(1-t)-\ln(t)\big)\mathrm{d}t$$

$$=\int_{\frac{1}{2}}^1\left(\ln^2(t)\ln(1-t)-\ln^3(t)\right)\left(\frac{1}{1-t}+\frac{1}{t}\right)\mathrm{d}t$$

$$=\int_{\frac{1}{2}}^1\frac{\ln^2(t)\ln(1-t)}{1-t}\mathrm{d}t+\int_{\frac{1}{2}}^1\frac{\ln^2(t)\ln(1-t)}{t}\mathrm{d}t-\int_{\frac{1}{2}}^1\frac{\ln^3(t)}{1-t}\mathrm{d}t-\int_{\frac{1}{2}}^1\frac{\ln^3(t)}{t}\mathrm{d}t$$

$$\underset{t\to1-t}{\smile}$$

$$\odot\int_{\frac{1}{2}}^1\frac{\ln^3(t)}{t}\mathrm{d}t=-\frac{1}{4}\ln^4(2)$$

$$\odot \int_{\frac{1}{2}}^{1} \frac{\ln^3(t)}{1-t}\,dt \overset{IBPs}{=} [-\ln(1-t)\ln^3(t)]_{\frac{1}{2}}^{1} + 3\int_{\frac{1}{2}}^{1} \frac{\ln^2(t)\ln(1-t)}{t}\,dt$$

$$= \ln^4(2) + 3\int_{\frac{1}{2}}^{1} \frac{\ln^2(t)\ln(1-t)}{t}\,dt$$

$$\Rightarrow B_+ = \int_0^{\frac{1}{2}} \frac{\ln^2(1-t)\ln(t)}{t}\,dt - 2\int_{\frac{1}{2}}^{1} \frac{\ln^2(t)\ln(1-t)}{t}\,dt - \frac{3}{4}\ln^4(2)$$

$$\odot \int_{\frac{1}{2}}^{1} \frac{\ln^2(t)\ln(1-t)}{t}\,dt = -\sum_{n=1}^{\infty} \frac{1}{n}\int_{\frac{1}{2}}^{1} t^{n-1}\ln^2(t)\,dt$$

$$= -\sum_{n=1}^{\infty} \frac{1}{n}\left(-\frac{2}{2^n n^3} + \frac{2}{n^3} - \frac{2\ln(2)}{2^n n^2} - \frac{\ln^2(2)}{2^n n}\right)$$

$$= 2\mathrm{Li}_4\left(\frac{1}{2}\right) - 2\zeta(4) + 2\ln(2)\mathrm{Li}_3\left(\frac{1}{2}\right) + \ln^2(2)\mathrm{Li}_2\left(\frac{1}{2}\right)$$

$$= 2\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{7}{4}\ln(2)\zeta(3) - \frac{\pi^4}{45} - \frac{1}{6}\ln^4(2) - \frac{\pi^2}{12}\ln^2(2)$$

$$\odot \int_0^{\frac{1}{2}} \frac{\ln^2(1-t)\ln(t)}{t}\,dt \overset{IBPs}{=} \left[\frac{1}{2}\ln^2(t)\ln^2(1-t)\right]_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} \frac{\ln^2(t)\ln(1-t)}{1-t}\,dt$$

$$= \frac{1}{2}\ln^4(2) + \int_0^{\frac{1}{2}} \frac{\ln^2(t)\ln(1-t)}{1-t}\,dt$$

$$= \frac{1}{2}\ln^4(2) + \int_0^{1} \frac{\ln^2(t)\ln(1-t)}{1-t}\,dt - \int_{\frac{1}{2}}^{1} \frac{\ln^2(t)\ln(1-t)}{1-t}\,dt = \frac{1}{2}\left(\frac{1}{2}\ln^4(2) + B_-\right)$$

$$= \frac{1}{4}\ln^4(2) - \frac{\pi^4}{360} \Rightarrow$$

$$B_+ = \frac{1}{4}\ln^4(2) - \frac{\pi^4}{360} - 2\left(2\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{7}{4}\ln(2)\zeta(3) - \frac{\pi^4}{45} - \frac{1}{6}\ln^4(2) - \frac{\pi^2}{12}\ln^2(2)\right)$$
$$- \frac{3}{4}\ln^4(2)$$

$$= -4\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{7}{2}\ln(2)\zeta(3) + \frac{\pi^4}{24} - \frac{1}{6}\ln^4(2) + \frac{\pi^2}{6}\ln^2(2)$$

$$\Rightarrow Q = \frac{1}{4}(A + B_+ - B_-) = -\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{23\pi^4}{5760} - \frac{1}{24}\ln^4(2) + \frac{\pi^2}{24}\ln^2(2)$$

$$\Rightarrow I(0) = \frac{\pi^4}{48} + 2Q = -2\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{83\pi^4}{2880} - \frac{1}{12}\ln^4(2) + \frac{\pi^2}{12}\ln^2(2)$$

3.) Conclusion

From section 1.) :

$$I(\alpha)=I(0)+\frac{\pi^2}{32}\left(\frac{\pi^2}{6}-\psi^{(1)}\left(1+\frac{\alpha}{2}\right)-\left(\gamma+\psi\left(1+\frac{\alpha}{2}\right)\right)^2\right)$$

From section 2.) :

$$I(0)=-2\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{83\pi^4}{2880}-\frac{1}{12}\ln^4(2)+\frac{\pi^2}{12}\ln^2(2)$$

$$\Rightarrow I(\alpha)=-2\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{49\pi^4}{1440}-\frac{1}{12}\ln^4(2)+\frac{\pi^2}{12}\ln^2(2)$$
$$-\frac{\pi^2}{32}\left(\psi^{(1)}\left(1+\frac{\alpha}{2}\right)+\left(\gamma+\psi^{(0)}\left(1+\frac{\alpha}{2}\right)\right)^2\right)$$

Solution 2 by: Yang Silva Cartolin-Peru

$$I(a)=\int_0^{\frac{\pi}{2}}\ln\left(\sin(x)\right)\tan(x)\left(\frac{\pi^2}{4}\sin^a(x)-x^2\right)\mathrm{d}x$$
$$-\int_0^{\frac{\pi}{2}}\ln\left(\cos(x)\right)\tan(x)\left(\frac{\pi^2}{4}\sin^a(x)-x^2\right)\mathrm{d}x$$

$$I(a)=i_1-i_2 \qquad (1)$$

$$\blacksquare i_1=\int_0^{\frac{\pi}{2}}\ln\left(\sin(x)\right)\tan(x)\left(\frac{\pi^2}{4}\sin^a(x)-x^2\right)\mathrm{d}x$$

$$\text{I. B. P:}\begin{cases}u=\frac{\pi^2}{4}\sin^a(x)-x^2\rightarrow \mathrm{d}u=\left(\frac{\pi^2 a}{4}\sin^{a-1}(x)\cos(x)-2x\right)\mathrm{d}x\\ \mathrm{d}v=\tan(x)\ln(\sin(x))\rightarrow v=\frac{1}{4}\mathrm{Li}_2\left(\cos^2(x)\right)\end{cases}$$

$$i_1=-\frac{1}{4}\int_0^{\frac{\pi}{2}}\mathrm{Li}_2\left(\cos^2(x)\right)\left(\frac{\pi^2 a}{4}\sin^{a-1}(x)\cos(x)-2x\right)\mathrm{d}x$$

$$i_1=-\frac{\pi^2 a}{16}\int_0^{\frac{\pi}{2}}\sin^{a-1}(x)\cos(x)\mathrm{Li}_2\left(\cos^2(x)\right)\mathrm{d}x+\frac{1}{2}\int_0^{\frac{\pi}{2}}x\,\mathrm{Li}_2\left(\cos^2(x)\right)\mathrm{d}x$$

$$i_1=-\frac{\pi^2 a}{16}\Delta_1+\frac{1}{2}\Delta_2 \qquad (2)$$

$$\hookrightarrow \Delta_1=\int_0^{\frac{\pi}{2}}\sin^{a-1}(x)\cos(x)\mathrm{Li}_2\left(\cos^2(x)\right)\mathrm{d}x$$

$$\overset{y=\sin^2(x)}{\Rightarrow} \Delta_1=\frac{1}{2}\int_0^1 y^{\frac{a}{2}-1}\mathrm{Li}_2(1-y)\,\mathrm{d}y,$$

$$\text{I. B. P:}\begin{cases} u=\text{Li}_2(1-y)\rightarrow du=\dfrac{\ln(y)}{1-y}\,dy \\ dv=y^{\frac{a}{2}-1}\,dy\rightarrow v=\dfrac{2}{a}y^{\frac{a}{2}} \end{cases}$$

$$\Delta_1=-\frac{1}{a}\int_0^1\frac{y^{\frac{a}{2}}\ln(y)}{1-y}\,dy$$

$*$ We know that: $\int_0^1\frac{y^b\ln(y)}{1-y}\,dy=-\psi^{(1)}(b+1)\Rightarrow\Delta_1=\frac{1}{a}\psi^{(1)}\left(\frac{a}{2}+1\right)\ \therefore$

$$\downarrow\Delta_2=\int_0^{\frac{\pi}{2}}x\,\text{Li}_2\left(\cos^2(x)\right)dx$$

*Fourier Series:

$$\text{Li}_2\left(\cos^2(x)\right)=\frac{\pi^2}{6}-2\ln^2(2)$$
$$-2\sum_{n=1}^{\infty}(-1)^n\left(\frac{2\ln(2)}{n}-\frac{(-1)^n}{n^2}-2\frac{\overline{H}_n}{n}\right)\cos(2nx),x\in\mathbb{R}$$

$$\Rightarrow\Delta_2=\left(\frac{\pi^2}{6}-2\ln^2(2)\right)\int_0^{\frac{\pi}{2}}x\,dx$$
$$-2\sum_{n=1}^{\infty}(-1)^n\left(\frac{2\ln(2)}{n}-\frac{(-1)^n}{n^2}-2\frac{\overline{H}_n}{n}\right)\int_0^{\frac{\pi}{2}}x\cos(2nx)\,dx$$

$$\Delta_2=\frac{\pi^4}{48}-\frac{\pi^2}{4}\ln(2)-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\left(\frac{2\ln(2)}{n}-\frac{(-1)^n}{n^2}-2\frac{\overline{H}_n}{n}\right)\left(\frac{(-1)^n-1}{n^2}\right)$$

$$\overset{n=2k+1}{\Rightarrow}\Delta_2=\frac{\pi^4}{48}-\frac{\pi^2}{4}\ln(2)-\sum_{k=0}^{\infty}\left(\frac{2\ln(2)}{(2k+1)^3}+\frac{1}{(2k+1)^4}-2\frac{\overline{H}_{2k+1}}{(2k+1)^3}\right)$$

$$\Delta_2=\frac{\pi^4}{48}-\frac{\pi^2}{4}\ln(2)-2\ln(2)\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}-\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}+2\sum_{k=0}^{\infty}\frac{H_{2k+1}-H_k}{(2k+1)^3}$$

$$\Delta_2=\frac{\pi^4}{96}-\frac{\pi^2}{4}\ln(2)-2\ln(2)\left(\frac{7\zeta(3)}{8}\right)$$
$$+2\left(-\text{Li}_4\left(\frac{1}{2}\right)+\frac{7}{8}\zeta(3)\ln(2)+\frac{19\pi^4}{2880}-\frac{1}{24}\ln^4(2)+\frac{\pi^2}{24}\ln^2(2)\right)$$

$$\Delta_2=\frac{17\pi^4}{720}-\frac{\pi^2}{6}\ln^2(2)-\frac{1}{12}\ln^4(2)-2\text{Li}_4\left(\frac{1}{2}\right)\ \therefore$$

$$\text{Therefore, in (2): } i_1 = -\frac{\pi^2}{16}\psi^{(1)}\left(\frac{a}{2}+1\right) + \frac{1}{2}\left(\frac{17\pi^4}{720} - \frac{\pi^2}{6}\ln^2(2) - \frac{1}{12}\ln^4(2) - 2\text{Li}_4\left(\frac{1}{2}\right)\right)$$

$$i_1 = -\frac{\pi^2}{16}\psi^{(1)}\left(\frac{a}{2}+1\right) + \frac{17\pi^4}{1440} - \frac{\pi^2}{12}\ln^2(2) - \frac{1}{24}\ln^4(2) - \text{Li}_4\left(\frac{1}{2}\right) \therefore$$

$$\blacksquare i_2 = \int_0^{\frac{\pi}{2}} \ln\left(\cos(x)\right)\tan(x)\left(\frac{\pi^2}{4}\sin^a(x) - x^2\right) dx$$

$$\text{I. B. P: } \begin{cases} u = \frac{\pi^2}{4}\sin^a(x) - x^2 \to du = \left(\frac{\pi^2 a}{4}\sin^{a-1}(x)\cos(x) - 2x\right) dx \\ dv = \tan(x)\ln\left(\cos(x)\right) \to v = -\frac{1}{2}\ln^2\left(\cos(x)\right) \end{cases}$$

$$i_2 = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln^2\left(\cos(x)\right)\left(\frac{\pi^2 a}{4}\sin^{a-1}(x)\cos(x) - 2x\right) dx$$

$$i_2 = \frac{\pi^2 a}{8}\int_0^{\frac{\pi}{2}} \ln^2\left(\cos(x)\right)\sin^{a-1}(x)\cos(x)\, dx - \int_0^{\frac{\pi}{2}} x\ln^2\left(\cos(x)\right) dx$$

$$i_2 = \frac{\pi^2 a}{8}\xi_1 - \xi_2 \qquad (3)$$

$$\text{↳ } \xi_1 = \int_0^{\frac{\pi}{2}} \ln^2\left(\cos(x)\right)\sin^{a-1}(x)\cos(x)\, dx$$

$$* \text{ Euler's Beta Function: } \text{B}(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}(x)\cos^{2n-1}(x)\, dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\frac{\partial \text{B}(m,n)}{\partial n} = 4\int_0^{\frac{\pi}{2}} \sin^{2m-1}(x)\cos^{2n-1}(x)\ln\left(\cos(x)\right) dx$$

$$\frac{\partial^2 \text{B}(m,n)}{\partial n^2} = 8\int_0^{\frac{\pi}{2}} \sin^{2m-1}(x)\cos^{2n-1}(x)\ln^2\left(\cos(x)\right) dx$$

$$\Rightarrow \xi_1 = \frac{1}{8}\left[\frac{\partial^2 \text{B}(m,n)}{\partial n^2}\right]_{m=\frac{a}{2}, n=1} = \frac{1}{8}\left[\frac{\partial^2}{\partial n^2}\left(\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right)\right]_{m=\frac{a}{2}, n=1}$$

$$\xi_1=\frac{1}{8}\left[\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\left(-2\psi^{(0)}(n)\psi^{(0)}(n+m)+\left(\psi^{(0)}(n+m)\right)^2-\psi^{(1)}(n+m)\right.\right.$$
$$\left.\left.+\left(\psi^{(0)}(n)\right)^2+\psi^{(1)}(n)\right)\right]_{m=\frac{a}{2},n=1}$$

$$\xi_1=\frac{\Gamma\left(\frac{a}{2}\right)}{8\Gamma\left(\frac{a}{2}+1\right)}\left(\left(\psi^{(0)}\left(\frac{a}{2}+1\right)\right)^2+2\gamma\psi^{(0)}\left(\frac{a}{2}+1\right)-\psi^{(1)}\left(\frac{a}{2}+1\right)+\frac{\pi^2}{6}+\gamma^2\right)$$

$$\overset{\Gamma\left(\frac{a}{2}+1\right)=\frac{a}{2}\Gamma\left(\frac{a}{2}\right)}{\Rightarrow}\xi_1=\frac{1}{4a}\left(\left(\psi^{(0)}\left(\frac{a}{2}+1\right)\right)^2+2\gamma\psi^{(0)}\left(\frac{a}{2}+1\right)-\psi^{(1)}\left(\frac{a}{2}+1\right)+\frac{\pi^2}{6}+\gamma^2\right)\therefore$$

$$\downarrow\xi_2=\int_0^{\frac{\pi}{2}}x\ln^2\left(\cos(x)\right)dx$$

*Fourier Series:

$$\ln^2\left(\cos(x)\right)=\ln^2(2)+\frac{\pi^2}{12}+2\sum_{n=1}^{\infty}(-1)^n\left(\frac{\ln(2)}{n}-\frac{1}{2n^2}+\frac{H_n}{n}\right)\cos(2nx),-\frac{\pi}{2}$$
$$<x<\frac{\pi}{2}$$

$$\Rightarrow\xi_2=\left(\ln^2(2)+\frac{\pi^2}{12}\right)\int_0^{\frac{\pi}{2}}x\ dx$$
$$+2\sum_{n=1}^{\infty}(-1)^n\left(\frac{\ln(2)}{n}-\frac{1}{2n^2}+\frac{H_n}{n}\right)\int_0^{\frac{\pi}{2}}x\cos(2nx)\,dx$$

$$\xi_2=\frac{\pi^2}{8}\ln^2(2)+\frac{\pi^4}{96}+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\left(\frac{\ln(2)}{n}-\frac{1}{2n^2}+\frac{H_n}{n}\right)\left(\frac{(-1)^n-1}{n^2}\right)$$

$$\overset{n=2k+1}{\Rightarrow}\xi_2=\frac{\pi^2}{8}\ln^2(2)+\frac{\pi^4}{96}+\sum_{k=0}^{\infty}\left(\frac{\ln(2)}{(2k+1)^3}-\frac{1}{2(2k+1)^4}-2\frac{H_{2k+1}}{(2k+1)^3}\right)$$

$$\xi_2=\frac{\pi^2}{8}\ln^2(2)+\frac{\pi^4}{96}+\ln(2)\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}-\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}-2\sum_{k=0}^{\infty}\frac{H_{2k+1}}{(2k+1)^3}$$

$$\xi_2=\frac{\pi^2}{8}\ln^2(2)+\frac{\pi^4}{96}+\ln(2)\left(\frac{7\zeta(3)}{8}\right)-\frac{1}{2}\left(\frac{\pi^4}{96}\right)$$
$$+\left(-\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{7}{8}\zeta(3)\ln(2)+\frac{\pi^4}{45}-\frac{1}{24}\ln^4(2)+\frac{\pi^2}{24}\ln^2(2)\right)$$

$$\xi_2=\frac{79\pi^4}{2880}+\frac{\pi^2}{6}\ln^2(2)-\frac{1}{24}\ln^4(2)-\mathrm{Li}_4\left(\frac{1}{2}\right)\ \therefore$$

Therefore, in (3):

$$i_2 = \frac{\pi^2}{32}\left(\left(\psi^{(0)}\left(\frac{a}{2}+1\right)\right)^2 + 2\gamma\psi^{(0)}\left(\frac{a}{2}+1\right) - \psi^{(1)}\left(\frac{a}{2}+1\right) + \frac{\pi^2}{6} + \gamma^2\right) - \frac{79\pi^4}{2880} - \frac{\pi^2}{6}\ln^2(2) + \frac{1}{24}\ln^4(2) + \mathrm{Li}_4\left(\frac{1}{2}\right) \quad \therefore$$

$$\text{Thus: } I(a) = \left(-\frac{\pi^2}{16}\psi^{(1)}\left(\frac{a}{2}+1\right) + \frac{17\pi^4}{1440} - \frac{\pi^2}{12}\ln^2(2) - \frac{1}{24}\ln^4(2) - \mathrm{Li}_4\left(\frac{1}{2}\right)\right) - \left(\begin{array}{l}\frac{\pi^2}{32}\left(\left(\psi^{(0)}\left(\frac{a}{2}+1\right)\right)^2 + 2\gamma\psi^{(0)}\left(\frac{a}{2}+1\right) - \psi^{(1)}\left(\frac{a}{2}+1\right) + \frac{\pi^2}{6} + \gamma^2\right) \\ \quad - \frac{79\pi^4}{2880} - \frac{\pi^2}{6}\ln^2(2) + \frac{1}{24}\ln^4(2) + \mathrm{Li}_4\left(\frac{1}{2}\right)\end{array}\right)$$

$$I(a) = -\frac{\pi^2}{32}\left[\left(\psi^{(0)}\left(\frac{a}{2}+1\right) + \gamma\right)^2 + \psi^{(1)}\left(\frac{a}{2}+1\right)\right] + \frac{49\pi^4}{1440} - \frac{1}{12}\ln^4(2) - 2\mathrm{Li}_4\left(\frac{1}{2}\right)$$

04 Prove that:

$$I=\int_0^1\int_0^1\frac{\ln\left(\frac{1+x^2y^2}{\sqrt{(1+xy)^3}}\right)\arctan(xy)}{(1+x^2y^2)\ln((xy)^{xy})}\mathrm{d}x\mathrm{d}y=\frac{5}{8}G\ln(2)-\frac{3\pi}{32}\ln^2(2)-\frac{\pi^3}{768}$$

$$G=\beta(2)\ is\ Catalan's\ constant$$

Proposed by: Samuel Adekunle Ajibade- Nigeria

Solution 1 by: Pham Duc Nam- Vietnam

$$I=\int_0^1\int_0^1\frac{\ln\left(\frac{1+x^2y^2}{\sqrt{(1+xy)^3}}\right)\arctan(xy)}{(1+x^2y^2)\ln((xy)^{xy})}\mathrm{d}x\mathrm{d}y=\frac{5}{8}G\ln(2)-\frac{3\pi}{32}\ln^2(2)-\frac{\pi^3}{768}$$

1.) Reduce the given integral

Using symmetry integral : $\int_0^1\int_0^1 f(xy)\mathrm{d}x\mathrm{d}y=-\int_0^1\ln(x)f(x)\mathrm{d}x$

$$\Rightarrow I=-\int_0^1\ln(t)\frac{\ln\left(\frac{1+t^2}{\sqrt{(1+t)^3}}\right)\arctan(t)}{t(1+t^2)\ln(t)}\mathrm{d}t$$

$$=\int_0^1\frac{\arctan(t)}{t(1+t^2)}\left(\frac{3}{2}\ln(1+t)-\ln(1+t^2)\right)\mathrm{d}t$$

Set : $a=\frac{\pi}{4}, l=\ln(2)$. With $t=\tan(\theta)$, we have : $\frac{\arctan(t)}{t(1+t^2)}\mathrm{d}t=\theta\cot(\theta)\mathrm{d}\theta$

$\ln\left(1+\tan^2(\theta)\right)=-2\ln\left(\cos(\theta)\right)$. Moreover, $1+\tan(\theta)=\frac{\sqrt{2}\cos(a-\theta)}{\cos(\theta)}$, and hence

$$\frac{3}{2}\ln\left(1+\tan(\theta)\right)-\ln\left(1+\tan^2(\theta)\right)=\frac{3l}{4}+\frac{1}{2}\ln\left(\cos(\theta)\right)+\frac{3}{2}\ln\left(\cos(a-\theta)\right)$$

Introduce 5 auxiliary integrals : $W=\int_0^a\theta\cot(\theta)\mathrm{d}\theta, C=\int_0^a\ln\left(\cos(\theta)\right)\mathrm{d}\theta=\frac{G}{2}-\frac{\pi}{4}\ln(2)$

$$P=\int_0^a\theta\cot(\theta)\ln\left(\cos(\theta)\right)\mathrm{d}\theta, Q=\int_0^a\theta\cot(\theta)\ln\left(\cos(a-\theta)\right)\mathrm{d}\theta, J$$

$$=\int_0^a\ln^2\left(\cos(\theta)\right)\mathrm{d}\theta\Rightarrow I=\frac{3}{4}lW+\frac{1}{2}P+\frac{3}{2}Q$$

2.) First relation $P+Q$. We will prove : $P+Q=\frac{a^3}{6}+\ln\left(\cos(a)\right)(W+C)-J$

Starting from : $\ln\big(\cos(a-\theta)\big) = -\int_{\theta}^{a}\tan(s-\theta)\mathrm{d}s \Rightarrow Q = -\int_{0}^{a}\int_{\theta}^{a}\theta\cot(\theta)\tan(s-\theta)\mathrm{d}s\mathrm{d}\theta$

$$\overset{\text{Change order of integration}}{=} -\int_{0}^{a}\left(\int_{0}^{s}\theta\cot(\theta)\tan(s-\theta)\mathrm{d}\theta\right)\mathrm{d}s$$

Using : $\cot(\theta)\tan(s-\theta) = \tan(s)\cot(\theta) - 1 - \tan(s)\tan(s-\theta)$ *(Blue relation)*

Define : $W(s) = \int_{0}^{s}u\cot(u)\mathrm{d}u, C(s) = \int_{0}^{s}\ln\big(\cos(u)\big)\mathrm{d}u$, also :

$$\int_{0}^{s}\theta\tan(s-\theta)\mathrm{d}\theta \overset{u=s-\theta}{=} \int_{0}^{s}(s-u)\tan(u)\mathrm{d}u \overset{\text{IBPs}}{=} -\int_{0}^{s}\ln\big(\cos(u)\big)\mathrm{d}u$$
$$= -C(s) \quad \text{(Red relation)}$$

Multiply both sides of Blue relation by θ, integrating from 0 to s and using Red

$$\Rightarrow \int_{0}^{s}\theta\cot(\theta)\tan(s-\theta)\mathrm{d}\theta = \tan(s)\big(W(s)+C(s)\big) - \frac{s^2}{2}$$

Therefore : $Q = \frac{a^3}{6} - \int_{0}^{a}\tan(s)\big(W(s)+C(s)\big)\mathrm{d}s$

$$\overset{\text{IBPs}}{=} \frac{a^3}{6} + \ln\big(\cos(a)\big)\big(W(a)+C(a)\big) - \int_{0}^{a}\ln\big(\cos(s)\big)\big(W'(s)+C'(s)\big)\mathrm{d}s$$

Since : $W'(s) = s\cot(s), C'(s) = \ln\big(\cos(s)\big)$, followed by definitions of above auxiliary integrals

We have : $P + Q = \frac{a^3}{6} + \ln\big(\cos(a)\big)(W+C) - J$, proved.

3.) <u>Second relation :</u> $P + \frac{3}{2}J$

We will prove : $P + \frac{3}{2}J = -2lC - \frac{7al^2}{8} + \frac{a^3}{3}$

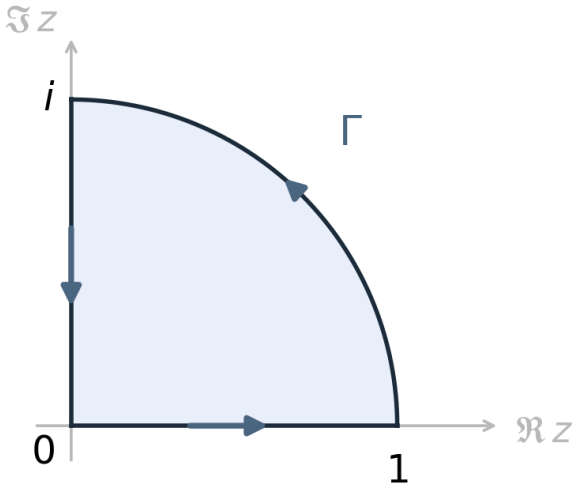
Define another integral : $V = \int_{0}^{1}\frac{\arctan(t)\ln(1+t^2)}{t}\mathrm{d}t$

Since $\leq t \leq 1$: $\mathrm{Log}(1+it) = \frac{1}{2}\ln(1+t^2) + i\arctan(t)$

Where : $\mathrm{Log}(.)$ denotes principal logarithm. Then : $\Im\{\mathrm{Log}^2(1+it)\} = \arctan(t)\ln(1+t^2)$

Let : $\Phi(z) = \frac{\mathrm{Log}^2(1+z)}{z}$ and integrate this function over the contour D as shown,

which defined by $D = \{z: |z| \leq 1, \Re\{z\} \geq 0, \Im\{z\} \geq 0\}$



For z in a neighborhood of D, $1+z$ avoids the nonpositive real axis, so the principal logarithm is analytic. Also $\mathbf{Log}(1+z)=z+O(z^2)$, hence $\Phi(z)=z+O(z^2)$ at $z=0$: the apparent singulairy is removable. Cauchy's theorem therefore applies to the entire closed contour

Cauchy's theorem gives : $\int_0^1 \frac{\ln^2(1+t)}{t}\mathbf{d}t+\int_\Gamma \Phi(z)\mathbf{d}z-\int_0^1 \frac{\mathbf{Log}^2(1+it)}{t}\mathbf{d}t=0$

The first integral is purely real then taking imaginary part gives us : $V=\Im\left\{\int_\Gamma \Phi(z)\mathbf{d}z\right\}$

On : Let : $z=e^{2i\theta}(0\le\theta\le a)$, in this interval $\cos(\theta)>0$ and :

$$1+e^{2i\theta}=2\cos(\theta)e^{i\theta},\mathbf{Log}\left(1+e^{2i\theta}\right)=\ln\left(2\cos(\theta)\right)+i\theta,\frac{\mathbf{d}z}{z}=2i\mathbf{d}\theta$$

$$\Rightarrow V=\Im\left\{2i\int_0^a\left(\ln\left(2\cos(\theta)\right)+i\theta\right)^2\mathbf{d}\theta\right\}=2\int_0^a\left(\ln^2\left(2\cos(\theta)\right)-\theta^2\right)\mathbf{d}\theta$$

$$=2J+4lC+2al^2-\frac{2a^3}{3}$$

On the other hand, let : $t=\tan(\theta)$ in $V\Rightarrow V=-2\int_0^a\theta\left(\cot(\theta)+\tan(\theta)\right)\ln\left(\cos(\theta)\right)\mathbf{d}\theta$

$$=-2P-2\int_0^a\theta\tan(\theta)\ln\left(\cos(\theta)\right)\mathbf{d}\theta$$

Integration by parts gives us :

$$\int_0^a\theta\tan(\theta)\ln\left(\cos(\theta)\right)\mathbf{d}\theta=-\frac{a^2}{2}\ln^2\left(\cos(a)\right)+\frac{1}{2}\int_0^a\ln^2\left(\cos(\theta)\right)\mathbf{d}\theta$$

$$= -\frac{al^2}{8} + \frac{1}{2}J \Rightarrow V = -2P - J + \frac{al^2}{4}$$

Now compare two forms of : $2J + 4lC + 2al^2 - \frac{2a^3}{3} = -2P - J + \frac{al^2}{4}$

$$\Leftrightarrow 2P + 3J = -4lC - \frac{7al^2}{4} + \frac{2a^3}{3} \Leftrightarrow P + \frac{3}{2}J = -2lC - \frac{7al^2}{8} + \frac{a^3}{3}, \text{proved.}$$

4.) Magical cancellation

From section 1.) : $I = \frac{3}{4}lW + \frac{1}{2}P + \frac{3}{2}Q = \frac{3}{4}lW + \frac{3}{2}(P + Q) - P$

From section 2.) :

$$P + Q = \frac{a^3}{6} - \ln(\cos(a))(W + C) - J \overset{a=\frac{\pi}{4}, l=\ln(2)}{=} \frac{a^3}{6} - \frac{1}{2}l(W + C) - J$$

From section 3.) : $P + \frac{3}{2}J = -2lC - \frac{7al^2}{8} + \frac{a^3}{3}$

$$\Rightarrow I = \frac{3}{4}lW + \frac{3}{2}\left(\frac{a^3}{6} - \frac{1}{2}l(W + C) - J\right) - P, \text{ then } lW \text{ cancels out nicely}$$

$$\Rightarrow I = \frac{a^3}{4} - \frac{3}{4}lC - \left(P + \frac{3}{2}J\right) = \frac{a^3}{4} - \frac{3}{4}lC - \left(-2lC - \frac{7al^2}{8} + \frac{a^3}{3}\right)$$

$$= \frac{5}{4}lC + \frac{7al^2}{8} - \frac{a^3}{12} = \frac{5}{4}\ln(2)\left(\frac{G}{2} - \frac{\pi}{4}\ln(2)\right) + \frac{7}{8}\frac{\pi}{4}\ln^2(2) - \frac{1}{12}\left(\frac{\pi}{4}\right)^3$$

$$= \frac{5}{8}G\ln(2) - \frac{3\pi}{32}\ln^2(2) - \frac{\pi^3}{768}, \text{proved.}$$

Solution 2: Yang Silva Cartolin-Peru

$$\Phi = \int_0^1 \int_0^1 \frac{\ln\left(\frac{1 + x^2y^2}{\sqrt{(1 + xy)^3}}\right)\arctan(xy)}{(1 + x^2y^2)\ln((xy)^{xy})}\,dxdy = \frac{5}{8}G\ln(2) - \frac{3\pi}{32}\ln^2(2) - \frac{\pi^3}{768}$$

$$\Phi = \int_0^1 \int_0^1 f(xy)\,dx\,dy \overset{u=xy, v=y}{\Rightarrow} \Phi = -\int_0^1 f(u)\ln(u)\,du$$

$$\Phi = -\int_0^1 \frac{\ln\left(\frac{1 + u^2}{(1 + u)^{\frac{3}{2}}}\right)\arctan(u)}{u(1 + u^2)}\,du$$

$$\Phi=\int_0^1\frac{\left(\frac{3}{2}\ln(1+u)-\ln(1+u^2)\right)\arctan(u)}{u(1+u^2)}\,du$$

$$\Phi=\frac{3}{2}\int_0^1\frac{\ln(1+u)\arctan(u)}{u(1+u^2)}\,du-\int_0^1\frac{\ln(1+u^2)\arctan(u)}{u(1+u^2)}\,du=\frac{3}{2}i_1-i_2$$

$$\blacksquare i_1=\int_0^1\frac{\ln(1+u)\arctan(u)}{u(1+u^2)}\,du$$

$$=\int_0^1\frac{\ln(1+u)\arctan(u)}{u}\,du-\int_0^1\frac{u\ln(1+u)\arctan(u)}{u^2+1}\,du$$

$$i_1=\xi_1-\xi_2$$

$$\hookrightarrow\xi_1=\int_0^1\frac{\ln(1+u)\arctan(u)}{u}\,du,\text{I. B. P:}\begin{cases}f=\ln(1+u)\to\,df=\dfrac{du}{1+u}\\ dg=\dfrac{\arctan(u)}{u}\,du\to g=\mathrm{Ti}_2(u)\end{cases}$$

$$\xi_1=G\ln(2)-\int_0^1\frac{\mathrm{Ti}_2(u)}{1+u}\,du=G\ln(2)-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\int_0^1\frac{u^{2n+1}}{1+u}\,du$$

$$\text{We know that:}\int_0^1\frac{x^{2n+1}}{1+x}\,dx=H_{2n+1}-H_n-\ln(2)$$

$$\xi_1=2G\ln(2)-\sum_{n=0}^{\infty}\frac{(-1)^nH_{2n+1}}{(2n+1)^2}+\sum_{n=0}^{\infty}\frac{(-1)^nH_n}{(2n+1)^2}$$

Where:

$$\sum_{n=0}^{\infty}\frac{(-1)^nH_{2n+1}}{(2n+1)^2}=\frac{1}{128}\left(-64G\ln(2)-128\Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\}+7\pi^3+4\pi\ln^2(2)\right)$$

$$\sum_{n=0}^{\infty}\frac{(-1)^nH_n}{(2n+1)^2}=-G\ln(2)+2\Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\}-\frac{\pi^3}{64}-\frac{\pi}{16}\ln^2(2)$$

$$\text{Thus: }\xi_1=\frac{3}{2}G\ln(2)+3\Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\}-\frac{9\pi^3}{128}-\frac{3\pi}{32}\ln^2(2)\ \therefore$$

$$\hookrightarrow\xi_2=\int_0^1\frac{u\ln(1+u)\arctan(u)}{u^2+1}\,du$$

$$\text{We know that: }\frac{\arctan(x)}{1+x^2}=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}x^{2n-1}(2H_{2n}-H_n),|x|<1$$

$$\Rightarrow \xi_2 = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}(2H_{2n}-H_n)\int_0^1 u^{2n}\ln(1+u)\,du$$

We know that: $\int_0^1 x^{2n}\ln(1+x)\,dx = \frac{2\ln(2)}{2n+1} - \frac{1}{(2n+1)^2} + \frac{H_n}{2n+1} - \frac{H_{2n}}{2n+1}$

$$\Rightarrow \xi_2 = -2\ln(2)\sum_{n=1}^{\infty}\frac{(-1)^n\left(H_{2n}-\frac{1}{2}H_n\right)}{2n+1} + \sum_{n=1}^{\infty}\frac{(-1)^n\left(H_{2n}-\frac{1}{2}H_n\right)}{(2n+1)^2}$$
$$-\sum_{n=1}^{\infty}\frac{(-1)^n\left(H_{2n}H_n-\frac{1}{2}H_n^2\right)}{2n+1} + \sum_{n=1}^{\infty}\frac{(-1)^n\left(H_{2n}^2-\frac{1}{2}H_nH_{2n}\right)}{2n+1}$$

$$\xi_2 = -2\ln(2)\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{2n+1} + \ln(2)\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{2n+1} + \sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{(2n+1)^2} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{(2n+1)^2}$$
$$-\frac{3}{2}\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}H_n}{2n+1} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^nH_n^2}{2n+1} + \sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}^2}{2n+1}$$

Where:

$$\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{2n+1} = -\frac{\pi}{8}\ln(2) \ \wedge \ \sum_{n=1}^{\infty}\frac{(-1)^nH_n}{2n+1} = G - \frac{\pi}{2}\ln(2)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{(2n+1)^2} = \frac{1}{128}\left(-64G\ln(2) - 128\Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\} + 3\pi^3 + 4\pi\ln^2(2)\right)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}H_n}{2n+1} = \frac{G}{2}\ln(2) + 2\Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\} - \frac{\pi^3}{16} + \frac{3\pi}{16}\ln^2(2)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^nH_n^2}{2n+1} = 8\Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\} - \frac{3\pi^3}{16} + \frac{3\pi}{4}\ln^2(2)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}^2}{2n+1} = \frac{G}{2}\ln(2) + 2\Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\} - \frac{11\pi^3}{192}$$

$$\text{Thus: } \xi_2 = \frac{3}{4}G\ln(2) + \Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\} - \frac{5\pi^3}{192} - \frac{3\pi}{32}\ln^2(2) \ \therefore$$

$$\text{Therefore: } i_1 = \frac{3}{2}G\ln(2) + 3\Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\} - \frac{9\pi^3}{128} - \frac{3\pi}{32}\ln^2(2)$$
$$-\left(\frac{3}{4}G\ln(2) + \Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\} - \frac{5\pi^3}{192} - \frac{3\pi}{32}\ln^2(2)\right)$$

$$i_1=\frac{3}{4}G\ln(2)+2\Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\}-\frac{17\pi^3}{384}\ \therefore$$

$$\blacksquare i_2=\int_0^1\frac{\ln(1+u^2)\arctan(u)}{u(1+u^2)}\,du$$

$$i_2=\int_0^1\frac{\ln(1+u^2)\arctan(u)}{u}\,du-\int_0^1\frac{u\ln(1+u^2)\arctan(u)}{1+u^2}\,du=\lambda_1-\lambda_2$$

$$\hookrightarrow\lambda_1=-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 u^{2n-1}\arctan(u)\,du$$

$$\text{We know that: }\int_0^1 x^{2n-1}\arctan(x)\,dx=\frac{1}{2n}\left(\frac{\pi}{4}-\frac{\pi}{4}(-1)^n+(-1)^n\overline{O}_n\right)$$

$$\lambda_1=\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}+\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\overline{O}_n}{n^2}$$

Where:

$$*\ \text{Dirichlet Eta Function: }\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=(1-2^{1-s})\zeta(s)$$

$$*\sum_{n=1}^{\infty}\frac{\overline{O}_n}{n^2}=\frac{5\pi^3}{32}+\frac{\pi}{8}\ln^2(2)-2G\ln(2)-4\Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\}$$

$$\text{Thus: }\lambda_1=-\frac{3\pi^3}{32}-\frac{\pi}{16}\ln^2(2)+G\ln(2)+2\Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\}\ \therefore$$

$$\hookrightarrow\lambda_2$$

$$=\int_0^1\frac{u\ln(1+u^2)\arctan(u)}{1+u^2}\,du,\ \text{I. B. P: }\begin{cases}f=\arctan(u)\to df=\dfrac{du}{1+u^2}\\ dg=\dfrac{u\ln(1+u^2)}{1+u^2}\,du\to g=\dfrac{1}{4}\ln^2(1+u^2)\end{cases}$$

$$\lambda_2=\frac{\pi}{16}\ln^2(2)-\frac{1}{4}\int_0^1\frac{\ln^2(1+u^2)}{1+u^2}\,du$$

$$\overset{u=\tan(x)}{\Rightarrow}\lambda_2=\frac{\pi}{16}\ln^2(2)-\int_0^{\frac{\pi}{4}}\ln^2\left(\cos(x)\right)dx$$

$$\text{We know that: }\cos(x)=\frac{e^{ix}+e^{-ix}}{2}\to 2e^{ix}\cos(x)=1+e^{2ix}$$

$$\ln(2)+ix+\ln\left(\cos(x)\right)=\ln(1+e^{2ix})$$

$$\ln^2(2)-x^2+\ln^2\big(\cos(x)\big)+2i\ln(2)x+2\ln(2)\ln\big(\cos(x)\big)+2ix\ln\big(\cos(x)\big)$$
$$=\ln^2(1+e^{2ix})$$

$$\Re\{\ln^2\big(\cos(x)\big)\}=\Re\{\ln^2(1+e^{2ix})\}-\ln^2(2)+x^2-2\ln(2)\ln\big(\cos(x)\big)$$

$$\Rightarrow \lambda_2=\frac{5\pi}{16}\ln^2(2)-\Re\left\{\int_0^{\frac{\pi}{4}}\ln^2\left(1+e^{2ix}\right)dx\right\}-\frac{\pi^3}{192}+2\ln(2)\int_0^{\frac{\pi}{4}}\ln\big(\cos(x)\big)\,dx$$

We know that:

$$*\ \ln^2(1+x)=2\sum_{n=1}^{\infty}\frac{(-1)^nH_{n-1}}{n}x^n\overset{x=e^{2ix}}{\Rightarrow}\ln^2(1+e^{2ix})=2\sum_{n=1}^{\infty}\frac{(-1)^nH_{n-1}}{n}e^{2inx}$$

$$\Re\{\ln^2(1+e^{2ix})\}=2\sum_{n=1}^{\infty}\frac{(-1)^nH_{n-1}}{n}\cos(2nx)$$

$$*\ \ln\big(\cos(x)\big)=-\ln(2)-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos(2nx)$$

$$\Rightarrow \lambda_2=\frac{5\pi}{16}\ln^2(2)-\sum_{n=1}^{\infty}\frac{(-1)^nH_{n-1}}{n^2}\sin\left(\frac{\pi n}{2}\right)-\frac{\pi^3}{192}$$
$$+2\ln(2)\left(-\frac{\pi}{4}\ln(2)-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\left(\frac{\pi n}{2}\right)\right)$$

$$\overset{n=2k+1}{\Rightarrow}\lambda_2=\frac{5\pi}{16}\ln^2(2)+\sum_{k=0}^{\infty}\frac{(-1)^kH_{2k}}{(2k+1)^2}-\frac{\pi^3}{192}-\frac{\pi}{2}\ln^2(2)+\ln(2)G$$

Where:

$$\sum_{k=0}^{\infty}\frac{(-1)^kH_{2k}}{(2k+1)^2}=\frac{1}{128}\left(-64G\ln(2)-128\Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\}+3\pi^3+4\pi\ln^2(2)\right)$$

$$\text{Thus: }\lambda_2=\frac{1}{2}G\ln(2)-\Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\}+\frac{7\pi^3}{384}-\frac{5\pi}{32}\ln^2(2)\ \ \therefore$$

$$\text{Therefore: }i_2=-\frac{3\pi^3}{32}-\frac{\pi}{16}\ln^2(2)+G\ln(2)+2\Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\}$$
$$-\left(\frac{1}{2}G\ln(2)-\Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\}+\frac{7\pi^3}{384}-\frac{5\pi}{32}\ln^2(2)\right)$$

$$i_2=\frac{1}{2}G\ln(2)+3\Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\}-\frac{25\pi^3}{384}+\frac{3\pi}{32}\ln^2(2)\ \ \therefore$$

Finally:

$$I=\frac{3}{2}\left(\frac{3}{4}G\ln(2)+2\Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\}-\frac{17\pi^3}{384}\right)$$
$$-\left(\frac{1}{2}G\ln(2)+3\Im\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\}-\frac{25\pi^3}{384}+\frac{3\pi}{32}\ln^2(2)\right)$$

$$I=\frac{5}{8}G\ln(2)-\frac{3\pi}{32}\ln^2(2)-\frac{\pi^3}{768}\ \therefore$$

Solution 3 by: Emedi Abner Ngoy- DR Congo

$$I=\int_0^1\int_0^1\frac{\ln\left(\frac{1+x^2y^2}{\sqrt{(1+xy)^3}}\right)\tan^{-1}(xy)}{(1+x^2y^2)\ln((xy)^{xy})}\mathrm{d}x\mathrm{d}y=\frac{5}{8}G\ln(2)-\frac{3\pi}{32}\ln^2(2)-\frac{\pi^3}{768}$$

$$\text{Let xy = t}$$

$$I=-\int_0^1 f(t)\ln t\,dt$$

$$I=-\int_0^1\frac{\ln\left(\frac{1+t^2}{\sqrt{(1+t)^3}}\right)\tan^{-1}(t)\ln t}{(1+t^2)\,t\ln t}\,dt$$

$$I=\frac{3}{2}\int_0^1\frac{\ln(1+t)\tan^{-1}t}{t(1+t^2)}\,dt-\int_0^1\frac{\ln(1+t^2)\tan^{-1}t}{t(1+t^2)}\,dt$$

$$I_1=\int_0^1\frac{\ln(1+t)\tan^{-1}t}{t(1+t^2)}\,dt,\quad I_2=\int_0^1\frac{\ln(1+t^2)\tan^{-1}t}{t(1+t^2)}\,dt$$

$$I_1=\int_0^1\left(\frac{1}{t}-\frac{t}{1+t^2}\right)\ln(1+t)\tan^{-1}t\,dt$$

$$N=\int_0^1\frac{\tan^{-1}t\ln(1+t)}{t}\,dt=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1 t^{2n}\ln(1+t)\,dt$$

$$N=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left[\frac{2\ln2}{2n+1}-\frac{1}{(2n+1)^2}+\frac{H_n}{2n+1}-\frac{H_{2n}}{2n+1}\right]$$

$$S_1=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=G,\quad S_2=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}=\beta(3)$$

$$S_3=\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{(2n+1)^2}$$

$$\int_0^1 \frac{t^{2n}}{1+t}\,dt = H_n + \ln 2 - H_{2n+1} + \frac{1}{2n+1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \int_0^1 \frac{t^{2n}}{1+t}\,dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \left(H_n + \ln 2 - H_{2n+1} + \frac{1}{2n+1}\right)$$

$$\int_0^1 \frac{1}{1+t}\left(\frac{1}{t}\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)^2}\right) dt, \quad \sum_{n=0}^{\infty} (-1)^n a_{2n+1}$$

$$= \Im\left(\sum_{n=1}^{\infty} i^n a_n\right)$$

$$\int_0^1 \frac{1}{1+t}\left(\frac{1}{t}\Im\left(\sum_{n=1}^{\infty} \frac{i^n t^n}{n^2}\right)\right) dt = \Im\left(\int_0^1 \frac{\mathrm{Li}_2(it)}{(1+t)t}\,dt\right)$$

$$\Im\left(\int_0^1 \frac{1}{(t+1)}\left(\int_0^1 -\frac{i\ln x}{1-itx}\,dx\right) dt\right) = \int_0^1 \frac{1}{1+t}\left(\int_0^1 -\Im\left(\frac{i}{1-itx}\right)\ln t\right) dt = K$$

$$\Im\left(\frac{i}{1-itx}\right) = \frac{1}{1+x^2t^2}$$

$$K = -\int_0^1 \frac{1}{1+t}\left(\int_0^1 \frac{\ln x}{1+x^2t^2}\,dx\right) dt, \quad let\ xt = u$$

$$K = \int_0^1 \int_0^t \frac{\ln\left(\frac{t}{u}\right)}{t(1+t)(1+u^2)}\,du\,dt = \int_0^1 \frac{1}{1+u^2}\left(\int_u^1 \left(\frac{1}{t} - \frac{1}{t+1}\right)\ln\left(\frac{t}{u}\right) dt\right) du$$

$$K = \int_0^1 \frac{1}{1+u^2}\left(\int_u^1 \left(\frac{1}{t} - \frac{1}{1+t}\right)(\ln t - \ln u)\,dt\right) du$$

$$K = \int_0^1 \frac{1}{1+u^2}\left(\frac{1}{2}\ln^2 t - \ln u \ln t - \mathrm{Li}_2(-t) - \ln t \ln(t+1) + \ln u \ln(t+1)\right)\Big|_u^1$$

$$K = \int_0^1 \frac{1}{1+u^2}\left(\mathrm{Li}_2(-u) + \frac{1}{2}\ln^2 u + \ln(2)\ln u + \frac{\pi^2}{12}\right) du$$

$$K = \int_0^1 \frac{\mathrm{Li}_2(-u)}{1+u^2}\,du + \frac{1}{2}\int_0^1 \frac{\ln^2 u}{1+u^2}\,du$$

$$+\ln(2)\int_0^1 \frac{\ln u}{1+u^2}\,du+\frac{\pi^2}{12}\int_0^1 \frac{1}{1+u^2}\,du$$

$$L=\int_0^1 \frac{\mathrm{Li}_2(-u)}{1+u^2}\,du=\int_0^1 \frac{1}{1+u^2}\left(\int_0^1 \frac{u\ln t}{1+ut}\,dt\right)du$$

$$L=\int_0^1 \ln t\left(\int_0^1 \frac{u}{(1+u^2)(1+ut)}\,du\right)dt$$

$$L=\frac{\pi}{4}\int_0^1 \frac{t\ln t}{1+t^2}\,dt+\frac{\ln(2)}{2}\int_0^1 \frac{\ln t}{1+t^2}\,dt-\int_0^1 \frac{\ln t\ln(1+t)}{1+t^2}\,dt$$

$$L=-\frac{\pi^3}{192}-\frac{\ln(2)\,G}{2}-P$$

$$P\ \ \mathrm{cfrd}\left(\int_0^1\int_0^1 \frac{\ln\left((1+xy)\sqrt{(1+x^2y^2)^3}\right)}{1+x^2y^2}\,dx\,dy\right)$$

$$P=-3\Im\left(\mathrm{Li}_3(1-i)\right)-\frac{5\pi^3}{64}-\frac{3\pi}{16}\ln^2 2-2\ln(2)\,G$$

$$L=\frac{7\pi^3}{96}+\frac{3\pi}{16}\ln^2 2+\frac{3}{2}\ln(2)\,G+3\Im\left(\mathrm{Li}_3(1-i)\right)$$

$$K=L+\frac{1}{2}\int_0^1 \frac{\ln^2 u}{1+u^2}\,du+\ln(2)\int_0^1 \frac{\ln u}{1+u^2}\,du+\frac{\pi^2}{12}\int_0^1 \frac{1}{1+u^2}\,du$$

$$K=L+\beta(3)-\ln(2)\,G+\frac{\pi^3}{48}$$

$$K=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\left(H_n+\ln(2)-H_{2n+1}+\frac{1}{(2n+1)}\right)$$

$$L+\beta(3)-\ln(2)\,G+\frac{\pi^3}{48}=S_3+\ln(2)\,G-\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^2}+\beta(3)$$

$$S_3=L-2\ln(2)\,G+\frac{\pi^3}{48}+\Im\left(\sum_{n=1}^{\infty}\frac{i^n H_n}{n^2}\right)$$

$$S_3=L-2\ln(2)\,G+\frac{\pi^3}{48}+\Im(\mathrm{Li}_3(i)-\mathrm{Li}_3(1-i)$$

$$+\ln(1-i)\mathrm{Li}_2(1-i)+\frac{1}{2}\ln(i)\ln^2(1-i)+\zeta(3)\Big)$$

$$S_3 = L - 2\ln(2)\,G + \frac{\pi^3}{48} - \frac{\pi}{16}\ln^2(2) - \frac{1}{2}\ln(2)\,G - \Im\left(\mathrm{Li}_3(1-i)\right)$$

$$S_3 = \Im\left(3\mathrm{Li}_3(1-i) - \mathrm{Li}_3(1-i)\right) + \left(\frac{7\pi^3}{96} + \frac{\pi^3}{48}\right)$$

$$+\left(\frac{3}{2}\ln(2)\,G - \frac{1}{2}\ln(2)\,G - 2\ln(2)\,G\right)$$

$$+\left(\frac{3}{16}\pi\ln^2 2 - \frac{\pi}{16}\ln^2 2\right)$$

$$S_3 = 2\Im\left(\mathrm{Li}_3(1-i)\right) + \frac{3\pi^3}{32} + \frac{\pi\ln^2(2)}{8} - \ln(2)\,G$$

$$S_4 = \sum_{n=0}^{\infty} \frac{(-1)^n H_{2n}}{(2n+1)^2}, \quad H_{2n} = H_{2n+1} - \frac{1}{2n+1}$$

$$S_4 = \Im\left(\sum_{n=1}^{\infty} \frac{i^n H_n}{n^2}\right) - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$$

$$S_4 = \Im(\mathrm{Li}_3(i) - \mathrm{Li}_3(1-i) + \ln(1-i)\mathrm{Li}_2(1$$

$$-i) + \frac{1}{2}\ln(i)\ln^2(1-i) + \zeta(3)\Big) - \beta(3)$$

$$S_4 = -\Im\left(\mathrm{Li}_3(1-i)\right) - \frac{\pi}{16}\ln^2 2 - \frac{1}{2}\ln(2)\,G - \frac{\pi^3}{32}$$

$$N = [2\ln(2)S_1 - S_2 + S_3 - S_4]$$

$$N = 2\ln(2)\,G - \beta(3) + 2\Im\left(\mathrm{Li}_3(1-i)\right) + \frac{3\pi^3}{32} + \frac{\pi\ln^2 2}{8}$$

$$-\ln(2)\,G + \Im\left(\mathrm{Li}_3(1-i)\right) + \frac{\pi}{16}\ln^2 2 + \frac{1}{2}\ln(2)\,G + \beta(3)$$

$$N = \int_0^1 \frac{\ln(1+t)\tan^{-1}t}{t}\,dt = 3\Im\left(\mathrm{Li}_3(1-i)\right) + \frac{3\pi^3}{32} + \frac{3\pi}{16}\ln^2 2 + \frac{3G}{2}\ln(2)$$

$$M = \int_0^1 \frac{t\ln(1+t)\tan^{-1}t}{1+t^2}\,dt$$

$$\frac{\tan^{-1}t}{1+t^2} = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\,(H_n - 2H_{2n})t^{2n-1}$$

$$M=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\left(H_n-2H_{2n}\right)\int_0^1 t^{2n}\ln(1+t)\,dt$$

$$M=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\left(H_n\right.$$

$$\left.-2H_{2n}\right)\left[\frac{2\ln(2)}{2n+1}-\frac{1}{(2n+1)^2}+\frac{H_n}{(2n+1)}-\frac{H_{2n}}{(2n+1)}\right]$$

$$M=\frac{1}{2}[2\ln(2)A-B+C-3D-4\ln(2)E+2F+2Q]$$

$$A=\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{(2n+1)},\quad B=\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{(2n+1)^2}$$

$$C=\sum_{n=1}^{\infty}\frac{(-1)^nH_n^2}{(2n+1)},\quad D=\sum_{n=1}^{\infty}\frac{(-1)^nH_nH_{2n}}{(2n+1)}$$

$$E=\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{(2n+1)},\quad F=\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{(2n+1)^2}$$

$$Q=\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}^2}{(2n+1)},$$

$$A=\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{(2n+1)}=?$$

$$\int_0^1\frac{t^{2n}}{1+t}\,dt=H_n+\ln(2)-H_{2n+1}+\frac{1}{2n+1}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\int_0^1\frac{t^{2n}}{1+t}\,dt$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\left[H_n+\ln(2)-H_{2n+1}+\frac{1}{2n+1}\right]$$

$$\int_0^1\frac{1}{1+t}\Im\left(\frac{1}{t}\sum_{n=0}^{\infty}\frac{(-1)^nt^{2n+1}}{(2n+1)}\right)dt=A$$

$$+\frac{\pi\ln(2)}{4}-\Im\left(\sum_{n=1}^{\infty}\frac{i^nH_n}{n}\right)+G$$

$$\Im\left(\int_0^1 \frac{-\ln(1-it)}{(1+t)t}\,dt\right) = A + \frac{\pi}{4}\ln(2) - \Im\left(\mathrm{Li}_2(i) + \frac{\ln^2(1-i)}{2}\right) + G$$

$$\Im\left(-\mathrm{Li}_2(-i) - \mathrm{Li}_2(-i) - \ln(1-i)\ln(i) + \mathrm{Li}_2\left(\frac{1-i}{2}\right)\right) =$$

$$A + \frac{\pi}{4}\ln(2) - G + \frac{\pi\ln(2)}{8} + G$$

$$-\left(\frac{\pi\ln(2)}{8} - G\right) = A + \frac{3\pi\ln(2)}{8}$$

$$G - \frac{\pi}{2}\ln(2) = A$$

$$B = S_3 = 2\Im\left(\mathrm{Li}_3(1-i)\right) + \frac{3\pi^3}{32} + \frac{\pi\ln^2 2}{8} - \ln(2)\,G$$

$$C = \sum_{n=1}^{\infty} \frac{(-1)^n H_n^2}{(2n+1)}, \quad \frac{1}{2n+1} = \int_0^1 t^{2n}\,dt$$

$$\sum_{n\geq 0} H_n^2\, t^n = \frac{\ln^2(1-t) + \mathrm{Li}_2(t)}{1-t}, \quad t \to -t^2$$

$$C = \int_0^1 \frac{\ln^2(1+t^2) + \mathrm{Li}_2(-t^2)}{1+t^2}\,dt$$

$$U = \int_0^1 \frac{\ln^2(1+t^2)}{1+t^2}\,dt$$

$$\text{ou} \quad \ln^2\left(\frac{2t}{1+t^2}\right) = \left(\ln 2 + \ln t - \ln(1+t^2)\right)^2$$

$$\ln^2\left(\frac{2t}{1+t^2}\right) = \ln^2(1+t^2) + \ln^2 t + \ln^2 2$$

$$+2\ln(2)\ln(t) - 2\ln(2)\ln(1+t^2) - 2\ln(t)\ln(1+t^2)$$

$$\ln^2(1+t^2) = \ln^2\left(\frac{2t}{1+t^2}\right) - \ln^2 t - \ln^2 2$$

$$-2\ln(2)\ln(t) + 2\ln(2)\ln(1+t^2) + 2\ln(t)\ln(1+t^2)$$

$$a = \int_0^1 \frac{\ln^2\left(\frac{2t}{1+t^2}\right)}{1+t^2}\,dt, \quad let\ t = \tan x$$

$$a=\int_0^{\pi/4}\ln^2\left(\frac{2\tan x}{1+\tan^2 x}\right)dx=\int_0^{\pi/4}\ln^2\left(\sin(2x)\right)dx$$

$$a=\frac{1}{2}\int_0^{\pi/2}\ln^2\left(\sin x\right)dx=\frac{1}{2}\frac{\partial^2}{\partial s^2}\left[\int_0^{\pi/2}\sin^s x\,dx\right]_{s=0}$$

$$a=\frac{1}{4}\frac{\partial^2}{\partial s^2}\left[B\left(\frac{s+1}{2},\frac{1}{2}\right)\right]=\frac{1}{4}\left(\frac{\pi^3}{12}+\pi\ln^2 2\right)$$

$$a=\frac{\pi^3}{48}+\frac{\pi}{4}\ln^2 2$$

$$b=-\int_0^1\frac{\ln^2 t}{1+t^2}\,dt=-\frac{\beta(3)\,\Gamma(3)}{(-1)^{3-1}}=-\frac{\pi^3}{16}$$

$$c=-\ln^2 2\int_0^1\frac{1}{1+t^2}\,dt=-\pi\frac{\ln^2 2}{4}$$

$$d=-2\ln(2)\int_0^1\frac{\ln(t)}{1+t^2}\,dt=+2\ln(2)\beta(2)$$

$$e=2\ln(2)\int_0^1\frac{\ln(1+t^2)}{1+t^2}\,dt=2\ln(2)\int_0^{\pi/4}\ln\left(\sec^2 t\right)dt$$

$$e=2\ln(2)\left(\frac{\pi}{2}\ln(2)-G\right)=\pi\ln^2 2-2\ln(2)G$$

$$f=2\int_0^1\frac{\ln t\ln(1+t^2)}{1+t^2}\,dt,\quad \frac{\ln(1+t^2)}{1+t^2}=\sum_{n=0}^{\infty}(-1)^{n+1}H_n\,t^{2n}$$

$$f=2\sum_{n=0}^{\infty}(-1)^{n+1}H_n\int_0^1 t^{2n}\ln t\,dt$$

$$f=2\sum_{n=0}^{\infty}\frac{(-1)^n H_n}{(2n+1)^2}=2B=2S_3$$

$$f=4\Im\left(\mathrm{Li}_3(1-i)\right)+\frac{3\pi^3}{16}+\frac{\pi\ln^2 2}{4}-2\ln(2)G$$

$$U=a+b+c+d+e+f$$

$$U=\frac{\pi^3}{48}+\frac{\pi}{4}\ln^2 2-\frac{\pi^3}{16}-\frac{\pi\ln^2 2}{4}+2\ln(2)G+\pi\ln^2 2$$

$$-2\ln(2)G+4\Im\left(\mathrm{Li}_3(1-i)\right)+\frac{3\pi^3}{16}+\frac{\pi\ln^2 2}{4}-2\ln(2)G$$

$$U = 4\Im\left(\mathrm{Li}_3(1-i)\right) + \frac{5\pi\ln^2 2}{4} - 2\ln(2)G + \frac{7\pi^3}{48}$$

$$C = U + \int_0^1 \frac{\mathrm{Li}_2(-t^2)}{1+t^2}\,dt$$

$$R = \int_0^1 \frac{\mathrm{Li}_2(-t^2)}{1+t^2}\,dt, \quad u = \mathrm{Li}_2(-t^2), \quad du = -\frac{2\ln(1+t^2)}{t}\,dt$$

$$v = \tan^{-1}t$$

$$R = [\mathrm{Li}_2(-t^2)\tan^{-1}t]_0^1 + 2\int_0^1 \frac{\ln(1+t^2)\tan^{-1}t}{t}\,dt$$

$$R = -\frac{\pi^3}{48} + 2S, \quad S = \int_0^1 \frac{\ln(1+t^2)\tan^{-1}t}{t}\,dt$$

$$\ln(1+t^2)\tan^{-1}t = -2\sum_{n=1}^{\infty}(-1)^n \frac{H_{2n}}{(2n+1)}\,t^{2n+1}$$

$$S = -2\sum_{n=1}^{\infty}\frac{(-1)^n H_{2n}}{(2n+1)}\int_0^1 t^{2n}\,dt = -2\sum_{n=1}^{\infty}\frac{(-1)^n H_{2n}}{(2n+1)^2}$$

$$S = -2S_4 = 2\Im\left(\mathrm{Li}_3(1-i)\right) + \frac{\pi}{8}\ln^2 2 + \ln(2)G + \frac{\pi^3}{16}$$

$$R = -\frac{\pi^3}{48} + 4\Im\left(\mathrm{Li}_3(1-i)\right) + \frac{\pi}{4}\ln^2 2 + 2\ln(2)G + \frac{\pi^3}{8}$$

$$R = \frac{5\pi^3}{48} + 4\Im\left(\mathrm{Li}_3(1-i)\right) + 2\ln(2)G + \frac{\pi}{4}\ln^2 2$$

$$C = U + R$$

$$C = 8\Im\left(\mathrm{Li}_3(1-i)\right) + \left(\frac{7\pi^3}{48} + \frac{5\pi^3}{48}\right) + \left(\frac{\pi}{4}\ln^2 2 + \frac{5\pi\ln^2 2}{4}\right)$$

$$C = 8\Im\left(\mathrm{Li}_3(1-i)\right) + \frac{\pi^3}{4} + \frac{3\pi}{2}\ln^2 2$$

$$D = \sum_{n=1}^{\infty}\frac{(-1)^n H_n H_{2n}}{(2n+1)}$$

$$\ln(1+t^2)\tan^{-1}t=-2\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{(2n+1)}\,t^{2n+1}$$

$$\frac{1}{t(1+t)}\ln(1+t^2)\tan^{-1}t=\frac{1}{t(1+t)}\left(-2\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{(2n+1)}\,t^{2n+1}\right)$$

$$-\int_0^1\frac{\ln(1+t^2)\tan^{-1}t}{t(1+t)}\,dt=2\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{(2n+1)}\int_0^1\frac{t^{2n}}{1+t}\,dt$$

$$\int_0^1\frac{t^{2n}}{1+t}\,dt=H_n+\ln(2)-H_{2n}$$

$$-\int_0^1\frac{\ln(1+t^2)\tan^{-1}t}{t(1+t)}\,dt=2\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{(2n+1)}(H_n+\ln(2)-H_{2n})$$

$$-\int_0^1\frac{\ln(1+t^2)\tan^{-1}t}{t(1+t)}\,dt=2D+2\ln(2)E-2Q$$

$$D=\frac{1}{2}\left(-\int_0^1\frac{\ln(1+t^2)\tan^{-1}t}{t(1+t)}\,dt-2\ln(2)E+2Q\right)$$

$$T=\int_0^1\frac{\tan^{-1}t\ln(1+t^2)}{t(1+t)}\,dt$$

$$\frac{1}{2}T=\frac{1}{2}\int_0^1\frac{\tan^{-1}t\ln(1+t^2)}{t(1+t)}\,dt$$

$$\frac{1}{2}T=\int_0^1\tan^{-1}t\left(\int_t^1\frac{1}{(1+x)(1+tx)}\,dx\right)dt$$

$$\frac{1}{2}T=\int_0^1\tan^{-1}t\left(\left[\frac{-\ln(1+xt)+\ln(1+x)}{1-t}\right]_t^1\right)dt$$

$$\frac{1}{2}T=\int_0^1\frac{\tan^{-1}t}{1-t}\ln\left(\frac{2(1+t^2)}{(1+t)^2}\right)dt,\quad let\ t\to\frac{1-x}{1+x}$$

$$\frac{1}{2}T=\int_0^1\left(\frac{\pi}{4}-\tan^{-1}x\right)\frac{\ln(1+x^2)}{x(x+1)}\,dx$$

$$\frac{1}{2}T=\frac{\pi}{4}\int_0^1\frac{\ln(1+x^2)}{x(1+x)}\,dx-T$$

$$\frac{3}{2}T=\frac{\pi}{4}W,\quad W(s)=\int_0^1\frac{\ln(1+s^2x^2)}{x(1+x)}\,dx$$

$$W'(s)=\int_0^1\frac{2sx}{(1+x)(1+s^2x^2)}\,dx$$

$$W=\int_0^1\left(\frac{2\tan^{-1}(s)}{1+s^2}+\frac{s\ln(1+s^2)}{(1+s^2)}-\ln(2)\frac{2s}{1+s^2}\right)ds$$

$$W=\left[(\tan^{-1}s)^2+\frac{1}{4}\ln^2(1+s^2)-\ln(2)\ln(1+s^2)\right]_0^1$$

$$W=\frac{\pi^2}{16}-\frac{3}{4}\ln^2 2$$

$$\frac{3}{2}T=\frac{\pi}{4}\left(\frac{\pi^2}{16}-\frac{3}{4}\ln^2 2\right)\Rightarrow T=\frac{\pi^3}{96}-\frac{\pi}{8}\ln^2 2$$

$$D=-\frac{\pi^3}{192}+\frac{\pi}{16}\ln^2 2-\ln(2)E+Q$$

$$E=\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{(2n+1)},\quad H_{2n}=H_{2n+1}-\frac{1}{2n+1}$$

$$E=\sum_{n=0}^{\infty}\frac{(-1)^nH_{2n+1}}{(2n+1)}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$$

$$E=\Im\left(\sum_{n=1}^{\infty}\frac{i^nH_n}{n}\right)-G=G-\frac{\pi\ln(2)}{8}-G=-\frac{\pi\ln(2)}{8}$$

$$Q=\sum_{n=0}^{\infty}\frac{(-1)^nH_{2n}^2}{(2n+1)},\quad H_{2n}=H_{2n+1}-\frac{1}{2n+1}$$

$$Q=\sum_{n=0}^{\infty}\frac{(-1)^nH_{2n+1}^2}{(2n+1)}-2\sum_{n=0}^{\infty}\frac{(-1)^nH_{2n+1}}{(2n+1)^2}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}$$

$$Q=\Im\left(\sum_{n=1}^{\infty}\frac{i^nH_n^2}{n}\right)-2\Im\left(\sum_{n=1}^{\infty}\frac{i^nH_n}{n^2}\right)+\beta(3)$$

$$\mu=\Im\left(\sum_{n=1}^{\infty}\frac{i^nH_n^2}{n}\right)$$

$$=\Im\left(\mathrm{Li}_3(i)-\mathrm{Li}_2(i)\ln(1-i)-\frac{1}{3}\ln^3(1-i)\right)$$

$$\mu=\frac{\pi^3}{48}-\frac{G\ln(2)}{2}+\frac{\pi\ln^2 2}{16}$$

$$\nu=\Im\left(\sum_{n=1}^{\infty}\frac{i^n H_n}{n^2}\right)=\Im(\mathrm{Li}_3(i)-\mathrm{Li}_3(1-i)$$

$$+\mathrm{Li}_2(1-i)\ln(1-i)+\frac{1}{2}\ln^2(1-i)\ln(i)+\zeta(3)\Big)$$

$$\nu=-\Im\big(\mathrm{Li}_3(1-i)\big)-\frac{\pi}{16}\ln^2 2-\frac{1}{2}\ln(2)G$$

$$Q=\mu-2\nu+\beta(3)=\frac{G\ln(2)}{2}+2\Im\big(\mathrm{Li}_3(1-i)\big)+\frac{5\pi^3}{96}+\frac{3\pi\ln^2 2}{16}$$

$$D=-\frac{\pi^3}{192}+\frac{\pi}{16}\ln^2 2-\ln(2)E+Q$$

$$D=\frac{G\ln(2)}{2}+2\Im\big(\mathrm{Li}_3(1-i)\big)+\frac{3\pi^3}{64}+\frac{3\pi\ln^2 2}{8}$$

$$M=\frac{1}{2}(2\ln(2)A-B+C-3D-4\ln(2)E+2F+2Q)$$

$$M=\frac{1}{2}(\Im(-2\mathrm{Li}_3(1-i)+8\mathrm{Li}_3(1-i)-6\mathrm{Li}_3(1$$

$$-i)-2\mathrm{Li}_3(1-i)+4\mathrm{Li}_3(1-i)))+\frac{1}{2}(2+1$$

$$-\frac{3}{2}-1+1)G\ln(2)+\frac{1}{2}\Big(-1-\frac{1}{8}+\frac{3}{2}-\frac{9}{8}+\frac{1}{2}$$

$$-\frac{1}{8}+\frac{3}{8}\Big)\pi\ln^2 2+\frac{1}{2}\Big(-\frac{3}{32}+\frac{1}{4}-\frac{9}{64}-\frac{1}{16}+\frac{5}{48}\Big)\pi^3$$

$$M=\Im\big(\mathrm{Li}_3(1-i)\big)+\frac{3}{4}G\ln(2)+\frac{11}{384}\pi^3$$

$$N=3\Im\big(\mathrm{Li}_3(1-i)\big)+\frac{3}{2}G\ln(2)+\frac{3}{32}\pi^3+\frac{3\pi\ln^2(2)}{16}$$

$$I_1=N-M$$

$$I_1=2\Im\big(\mathrm{Li}_3(1-i)\big)+\frac{25\pi^3}{384}+\frac{3}{4}\ln(2)G+\frac{3\pi}{16}\ln^2 2$$

$$I_2=\int_0^1\frac{\ln(1+t^2)\tan^{-1}t}{t(1+t^2)}\,dt=\int_0^1\left(\frac{1}{t}-\frac{t}{1+t^2}\right)\ln(1+t^2)\tan^{-1}t\,dt$$

$$X=\int_0^1\frac{\ln(1+t^2)\tan^{-1}t}{t}\,dt,\quad Y=\int_0^1\frac{t\ln(1+t^2)\tan^{-1}t}{1+t^2}\,dt$$

$$X=-2\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{(2n+1)}\int_0^1 t^{2n}\,dt=-2\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{(2n+1)^2}$$

$$X=-2F=2\Im\left(\mathrm{Li}_3(1-i)\right)+\frac{\pi}{8}\ln^2 2+\ln(2)G+\frac{\pi^3}{16}$$

$$Y=\int_0^1\frac{t\ln(1+t^2)\tan^{-1}t}{(1+t^2)}\,dt\quad(\text{IPP})$$

$$u=\tan^{-1}t,\quad du=\frac{1}{1+t^2}\,dt,\quad v=\frac{1}{4}\int d\left(\ln^2(1+t^2)\right)$$

$$Y=\left[\frac{1}{4}\tan^{-1}t\ln^2(1+t^2)\right]_0^1-\frac{1}{4}\int_0^1\frac{\ln^2(1+t^2)}{1+t^2}\,dt$$

$$Y=\frac{\pi\ln^2 2}{16}-\frac{1}{4}U$$

$$Y=\frac{\pi\ln^2 2}{16}-\frac{1}{4}\left[4\Im\left(\mathrm{Li}_3(1-i)\right)+\frac{5\pi\ln^2 2}{4}-2\ln(2)G+\frac{7\pi^3}{48}\right]$$

$$Y=-\Im\left(\mathrm{Li}_3(1-i)\right)-\frac{7\pi^3}{192}-\frac{\pi\ln^2 2}{4}+\frac{G}{2}\ln(2)$$

$$I_2=X-Y=3\Im\left(\mathrm{Li}_3(1-i)\right)+\frac{19\pi^3}{192}+\frac{3}{8}\pi\ln^2 2+\frac{G}{2}\ln(2)$$

$$I=\frac{3}{2}I_1-I_2$$

$$I=\frac{3}{2}\left[2\Im\left(\mathrm{Li}_3(1-i)\right)+\frac{25\pi^3}{384}+\frac{3}{4}\ln(2)G+\frac{3\pi}{16}\ln^2 2\right]$$

$$-\left[3\Im\left(\mathrm{Li}_3(1-i)\right)+\frac{19\pi^3}{192}+\frac{3}{8}\pi\ln^2 2+\frac{G}{2}\ln(2)\right]$$

$$I=\left(\frac{25}{256}-\frac{19}{192}\right)\pi^3+\left(\frac{9}{8}-\frac{1}{2}\right)G\ln(2)+\left(\frac{9}{32}-\frac{3}{8}\right)\pi\ln^2 2$$

$$I=\frac{5}{8}G\ln(2)-\frac{3}{32}\pi\ln^2 2-\frac{\pi^3}{768}$$