ROMANIAN MATHEMATICAL MAGAZINE

SOLVED RMM-PROBLEMS BY OMKUMAR RAO

By Omkumar Rao-India

SP.610. $If\ a,b,c>0\ then:$

$$\sqrt{a^2+1}+\sqrt{b^2+1}+\sqrt{c^2+1}\geq\sqrt{(a+b+c)^2+9}$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$$\vec{m}=(a,1),\vec{n}=(b,1),\vec{q}=(c,1)$$

$$Let\ |\vec{m}|=\sqrt{a^2+1},|\vec{n}|=\sqrt{b^2+1}\ ,|\vec{q}|=\sqrt{c^2+1}$$

By Triangle Inequality of vectors:

$$|\vec{m}|+|\vec{n}|+|\vec{q}|\geq|\vec{m}+\vec{n}+\vec{q}|\rightarrow(1)$$

$$\vec{m}+\vec{n}+\vec{q}=(a+b+c,3)$$

$$|\vec{m}+\vec{n}+\vec{q}|=\sqrt{(a+b+c)^2+3^2}$$

$$From\ (1):$$

$$\sqrt{a^2+1}+\sqrt{b^2+1}+\sqrt{c^2+1}\geq\sqrt{(a+b+c)^2+9}$$

Equality holds if and only if the vectors $\vec{m},\vec{n},\vec{q}$ ***are collinear , that is ,***

$$\frac{a}{1}=\frac{b}{1}=\frac{c}{1}\Rightarrow a=b=c$$

SP.611. $If\ x,y,z>0\ then:$

$$(x^5+y^5+z^5)(x^6+y^6+z^6)(x^2+y^2+z^2)^5\geq(x^3+y^3+z^3)^7$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

Generalized weighted Power – mean (Holder) inequality: for any positive real numbers m_i,k_i,q_i ***and positive weights*** u,v,w ***satisfying*** $u+v+w=1$

$$\left(\sum_{i=1}^{n}m_i\right)^u\left(\sum_{i=1}^{n}k_i\right)^v\left(\sum_{i=1}^{n}q_i\right)^w\geq\sum_{i=1}^{n}m_i{}^uk_i{}^vq_i{}^w$$

$$(m_1,m_2,m_3)=(x^5,y^5,z^5),(k_1,k_2,k_3)=(x^6,y^6,z^6),(q_1,q_2,q_3)=(x^2,y^2,z^2)$$

$$Weights: u=\frac{1}{7}, v=\frac{1}{7}, w=\frac{5}{7}, u+v+w=1$$

$$\left(\sum x^5\right)^{\frac{1}{7}}\left(\sum x^6\right)^{\frac{1}{7}}\left(\sum x^2\right)^{\frac{5}{7}} \geq \sum x^{\frac{5}{7}+\frac{6}{7}+\frac{10}{7}}$$

$$\left(\sum x^5\right)^{\frac{1}{7}}\left(\sum x^6\right)^{\frac{1}{7}}\left(\sum x^2\right)^{\frac{5}{7}} \geq \sum x^3$$

$$\left(\sum x^5\right)\left(\sum x^6\right)\left(\sum x^2\right)^5 \geq \left(\sum x^3\right)^7$$

Hence Proved.

SP.594. *Find $x, y > 0$ such that:*

$$(\ln(xy))^2 = \ln(xe).\ln\left(\frac{y}{e}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$$Let\ \ln(x)=a, \ln(y)=b. Then\ (a+b)^2=(a+1)(b-1)$$
$$a^2+a(b+1)+(b^2-b+1)=0$$
$$For\ a\ to\ be\ real, the\ discriminant\ must\ be\ non-negative:$$
$$D=(b+1)^2-4(b^2-b+1)\geq 0, \qquad -3b^2+6b-3\geq 0$$
$$(b-1)^2\leq 0 \implies b=1, \qquad (a+1)^2=0\implies a=-1$$
$$\ln(x)=-1\implies x=\frac{1}{e}, \qquad \ln(y)=1\implies y=e$$

$$(x,y)=\left(\frac{1}{e}, e\right)$$

UP.593. *For $b \geq a$ show that:*

$$\int_a^b \frac{(x+1)^3-3x}{e^{x^3}}dx \leq b-a+\ln\left(\frac{b^3+1}{a^3+1}\right)$$

with equality if and only if $a=b$.

Proposed by Huseyin Yigit Emekci – Turkey

Solution by Omkumar Rao – India

$$For\ b\geq a, (x+1)^3-3x=x^3+3x^2+1, \qquad e^{x^3}\geq 1+x^3\ for\ all\ x^3\in R,$$
$$Let\ I=\int_a^b \frac{x^3+3x^2+1}{e^{x^3}}dx$$
$$\frac{1}{e^{x^3}}\leq\frac{1}{1+x^3}$$

$$\frac{x^3+3x^2+1}{e^{x^3}} \le \frac{x^3+3x^2+1}{x^3+1}$$

$$\int_a^b \frac{x^3+3x^2+1}{e^{x^3}}\,dx \le \int_a^b \left(1+\frac{3x^2}{x^3+1}\right)dx$$

$$I < b-a+\ln\left(\frac{b^3+1}{a^3+1}\right)$$

Equality holds if and only if $a=b$, *since* $0 \le 0$
Hence Proved

SP. 592. *If* $a,b,c>0$ *and* $n \in N, n \ge 2$ *then*:

$$\sum a\sqrt[n]{b^n+c^n} \ge \sqrt[n]{2}(ab+bc+ca)$$

Proposed by Marin Chirciu – Romania

Solution by Omkumar Rao – India

Applying Power mean Inequality for $n \ge 2$:

$$\left(\frac{b^n+c^n}{2}\right)^{\frac{1}{n}} \ge \frac{b+c}{2}, \qquad \sqrt[n]{b^n+c^n} \ge \sqrt[n]{2}\,\frac{b+c}{2}$$

$$\sum a\sqrt[n]{b^n+c^n} \ge \frac{\sqrt[n]{2}\,a(b+c)}{2}+\frac{\sqrt[n]{2}\,b(c+a)}{2}+\frac{\sqrt[n]{2}\,c(a+b)}{2}$$

$$\sum a\sqrt[n]{b^n+c^n} \ge \frac{\sqrt[n]{2}}{2}.2(ab+bc+ca)$$

$$\sum a\sqrt[n]{b^n+c^n} \ge \sqrt[n]{2}(ab+bc+ca)$$

Hence Proved.

SP. 607. *Let be the function* $f:[0,1]\to R$ *integrable such that*

$$f(1)=1 \text{ and } \int_x^y f(t)\,dt = \frac{1}{2}\big(yf(y)-xf(x)\big), \forall x,y \in [0,1].$$ *Find*:

$$I = \int_0^{\frac{\pi}{4}} f(x).(tanx)^2\,dx$$

Proposed by Marian Ursarescu, Florica Anastase – Romania

Solution by Omkumar Rao

Differentiating both sides with respect to y (treating x as constant):

$$f(y)=\frac{1}{2}\left(f(y)+yf'(y)\right),\qquad \int\frac{f'(y)}{f(y)}dy=\int\frac{1}{y}dy,\qquad f(y)=ky$$

Using $f(1)=1: f(y)=y$

$$I=\int_0^{\frac{\pi}{4}}x(tanx)^2\,dx,\qquad I=\int_0^{\frac{\pi}{4}}x(secx)^2\,dx-\int_0^{\frac{\pi}{4}}x\,dx\,,I=\frac{\pi}{4}-\int_0^{\frac{\pi}{4}}tanx\,dx-\frac{\pi^2}{32}$$

$$\boxed{I=\frac{\pi}{4}-\frac{\pi^2}{32}-\frac{1}{2}ln2}$$

UP.609. *If $f:[a,b]\longrightarrow(0,\infty)$ is a continuous function such*

$$f(a+b-x)+f(x)=c,\forall x\in[a,b],a+b=\frac{\pi}{2},$$

$$I=\int_a^b\frac{(\sin x)^n+f(x)+d}{(\sin x)^n+(\cos x)^n+c+2d}dx$$

where $n\in N*$ and $d\geq 0$.

Proposed by D.M.Batinetu – Giurgiu, Neculai Stanciu – Romania
Solution by Omkumar Rao – India

$$I=\int_a^b\frac{(\sin x)^n+f(x)+d}{(\sin x)^n+(\cos x)^n+c+2d}dx\longrightarrow(1)$$

$$x\longrightarrow a+b-x$$

$$f(a+b-x)+f(x)=c$$

$$I=\int_a^b\frac{(\sin x)^n+c-f(x)+d}{(\cos x)^n+(\sin x)^n+c+2d}dx\longrightarrow(2)$$

Adding (1) and (2):

$$2I=\int_a^b\frac{(\sin x)^n+(\cos x)^n+c+2d}{(\sin x)^n+(\cos x)^n+c+2d}dx$$

$$\boxed{I=\frac{b-a}{2}}$$

ROMANIAN MATHEMATICAL MAGAZINE

JP.611. *Find the angle between the real planes*:

$$P1: 2x + y + 3z - 1 = 0, \qquad P2: 3x - 2y + z + 1 = 0$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

The angle between two planes is determined by the angle between their normal vectors. For plane P1, the normal vector is $n_1 = (2, 1, 3)$ *and for plane P2, the normal vector is* $n_2 = (3, -2, 1)$.

The angle θ *between the planes is*:

$$\cos\theta = \frac{|n_1 . n_2|}{|n_1| . |n_2|}$$

$$\cos\theta = \frac{|7|}{\sqrt{14}\sqrt{14}} = \frac{7}{14} = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ = \frac{\pi}{3}\ radians$$

The angle between planes P1 and P2 is 60° *or* $\frac{\pi}{3}$ *radians*.

JP.586. *If* a, b, c *are sides of acute* ΔABC

$$\frac{2}{a^2 + c^2 - b^2} = \frac{1}{b^2 + c^2 - a^2} + \frac{1}{c^2 + a^2 - b^2}$$ *then* $(\tan B)^2 \geq \tan A . \tan C$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

By the Law of Cosines: $b^2 + c^2 - a^2 = 2bc\ \cos A$,

$a^2 + c^2 - b^2 = 2ac\ \cos B$, $a^2 + b^2 - c^2 = 2ab\ \cos C$

$$\frac{2}{2ac\cos B} = \frac{1}{2bc\ \cos A} + \frac{1}{2ab\ \cos C}$$

$$\frac{2b}{\cos B} = \frac{a}{\cos A} + \frac{c}{\cos C}$$

By law of Sines, $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$.

$$2\tan B = \tan A + \tan C$$

Since ΔABC *is acute*, $\tan A, \tan A, \tan C > 0$.

By AM – GM Inequality: $\frac{\tan A + \tan C}{2} \geq \sqrt{\tan A . \tan C}$

$$: 2\tan B = \tan A + \tan C$$

$$\tan B \geq \sqrt{\tan A . \tan C}$$

$(\tan B)^2 \geq \tan A . \tan C$

Hence Proved

ROMANIAN MATHEMATICAL MAGAZINE

SP.606. *Let be* $A(2,1,0); B(0,1,2); C(3,0,1); D(4,4,4)$. *Find the volume of the tetrahedron* $ABCD$.

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

The volume of tetrahedron ABCD is given by: $V = \frac{1}{6}\left|\left[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right]\right|$

$$\overrightarrow{AB} = \vec{B} - \vec{A} = (-2,0,2), \overrightarrow{AC} = \vec{C} - \vec{A} = (1,-1,1)$$

$$\overrightarrow{AD} = \vec{D} - \vec{A} = (2,3,4), \qquad V = \frac{1}{6}\begin{vmatrix} -2 & 0 & 2 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{vmatrix}$$

$$\begin{vmatrix} -2 & 0 & 2 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = -2[(-1)(4) - (1)(3)] - 0[(1)(4) - (1)(2)] + 2[(1)(3) - (-1)(2)]$$

$$= 24$$

$$\boxed{V = \frac{1}{6}(24) = 4 \text{ cubic units}}$$

UP.588. *If* $a \geq 0$ *then*:

$$15\left(\int_0^a \frac{x}{e^x}dx . \int_0^a \frac{x^2}{e^x}dx . \int_0^a \frac{x^3}{e^x}dx\right)^2 \leq a^9\left(\int_0^a \frac{x^2}{e^{2x}}dx\right)^3$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

Cauchy – Schwarz Inequality:

$$\left(\int_0^a \frac{x}{e^x}dx\right)^2 \leq \left(\int_0^a x^2e^{-2x}dx\right)\left(\int_0^a 1\,dx\right) = a\left(\int_0^a \frac{x^2}{e^{2x}}dx\right) \Rightarrow (1)$$

$$\left(\int_0^a \frac{x^2}{e^x}dx\right)^2 \leq \left(\int_0^a x^2e^{-2x}dx\right)\left(\int_0^a x^2\,dx\right) = \frac{a^3}{3}\left(\int_0^a \frac{x^2}{e^{2x}}dx\right) \Rightarrow (2)$$

$$\left(\int_0^a \frac{x^3}{e^x}dx\right)^2 \leq \left(\int_0^a x^2e^{-2x}dx\right)\left(\int_0^a x^4\,dx\right) = \frac{a^5}{5}\left(\int_0^a \frac{x^2}{e^{2x}}dx\right) \Rightarrow (3)$$

Multiplying these three nequalities:

$$\left(\int_0^a \frac{x}{e^x}dx . \int_0^a \frac{x^2}{e^x}dx . \int_0^a \frac{x^3}{e^x}dx\right)^2 \leq \left(a . \frac{a^3}{3} . \frac{a^5}{5}\right)\left(\int_0^a \frac{x^2}{e^x}dx\right)^3$$

$$15\left(\int_0^a \frac{x}{e^x}dx.\int_0^a \frac{x^2}{e^x}dx.\int_0^a \frac{x^3}{e^x}dx\right)^2 \le a^9\left(\int_0^a \frac{x^2}{e^{2x}}dx\right)^3$$

Hence Proved

SP.609. *If* $a,b,c>0$ *then*:

$$\sqrt{a^2b^2+c^2}+\sqrt{c^2b^2+a^2}+\sqrt{c^2a^2+b^2}\ge\sqrt{(a+b+c)^2+(ab+bc+ca)^2}$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$\vec{u}=(ab,c),\vec{v}=(bc,a),,\vec{w}=(ca,b)\in R^2$:

$|\vec{u}|=\sqrt{a^2b^2+c^2},|\vec{v}|=\sqrt{c^2b^2+a^2}$

$|\vec{w}|=\sqrt{c^2a^2+b^2}$

Minkowski Inequality : $|\vec{u}|+|\vec{v}|+|\vec{w}|\ge|\vec{u}+\vec{v}+\vec{w}|$

$\vec{u}+\vec{v}+\vec{w}=(ab+bc+ca,a+b+c)$:

$$\sqrt{a^2b^2+c^2}+\sqrt{c^2b^2+a^2}+\sqrt{c^2a^2+b^2}\ge\sqrt{(a+b+c)^2+(ab+bc+ca)^2}$$

$$\textbf{Equality}:\vec{u}\|\vec{v}\|\vec{w}\Leftrightarrow\frac{ab}{c}=\frac{bc}{a}=\frac{ca}{b}\Leftrightarrow a=b=c$$

Hence Proved

UP.611. *If* $R_+^*\to R_+^*$ *is a continuousfunction and* $(x_n)_{n\ge1}$,

$$x_n=\sum_{k=1}^{n}\frac{1}{k},\ find$$

$$\lim_{n\to\infty}\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx.$$

Proposed by D.M.Batinetu – Giurgiu, Neculai Stanciu – Romania

Solution by Omkumar Rao – India

$$I_n=\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx\ \to(1)$$

$$I_n\overset{x\to e^{x_n}+e^{x_{n+1}}-x}{=}\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(e^{x_{n+1}}-x)}{f(x-e^{x_n})+f(e^{x_{n+1}}-x)}dx\to(2)$$

$$2I_n = \int_{e^{x_n}}^{e^{x_{n+1}}} 1\,dx$$

$$I_n = \frac{e^{x_{n+1}} - e^{x_n}}{2}, \qquad e^{x_{n+1}} - e^{x_n} = e^{x_n}\left(e^{\frac{1}{n+1}} - 1\right)$$

$$e^{x_{n+1}} - e^{x_n} = e^{x_n}\left(\frac{1}{n+1} + O\left(\frac{1}{n^2}\right)\right), \qquad x_n = \ln n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)$$

where γ is Euler Mascheroni constant.

$$I_n = ne^{\gamma}\left(1 + O\left(\frac{1}{n}\right)\right)\left(\frac{1}{n+1}\right), \qquad I_n = \frac{e^{\gamma}}{2}\cdot\frac{n}{n+1}\cdot\left(1 + O\left(\frac{1}{n}\right)\right)$$

$$As\ n \to \infty, \frac{n}{n+1} \to 1, 1 + O\left(\frac{1}{n}\right) \to 1$$

$$\lim_{n\to\infty} \int_{e^{x_n}}^{e^{x_{n+1}}} \frac{f(x - e^{x_n})}{f(e^{x_{n+1}} - x) + f(x - e^{x_n})}\,dx = \frac{e^{\gamma}}{2}$$

UP.597. ***Find the following limit:***

$$L = \lim_{n\to\infty}\left(\frac{1}{2^n}\cdot\lim_{x\to\frac{\pi}{n}}\left(\sum_{k=0}^{n}\binom{n}{k}\sin\big((k+1)x\big)\right)\right)$$

Proposed by Marian Ursarescu, Florica Anastase – Romania

Solution by Omkumar Rao – India

$$S_n(x) = \sum_{k=0}^{n}\binom{n}{k}\sin\big((k+1)x\big).$$

$$S_n(x) = Im\left[\sum_{k=0}^{n}\binom{n}{k}e^{i(k+1)x}\right] = Im\left[e^{ix}\sum_{k=0}^{n}\binom{n}{k}e^{ikx}\right] = Im[e^{ix}(1+e^{ix})^n]$$

$$S_n(x) = 2^n\left(\cos\frac{x}{2}\right)^n \sin\left(x + \frac{nx}{2}\right)$$

$$x = \frac{\pi}{n}$$

$$\frac{1}{2^n}S_n\left(\frac{\pi}{n}\right) = \left(\cos\frac{\pi}{2n}\right)^n \sin\left(\frac{\pi}{n} + \frac{\pi}{2}\right)$$

$$\left(\cos\frac{\pi}{2n}\right)^n = \left(1 - \frac{\pi^2}{8n^2} + O\left(\frac{1}{n^4}\right)\right)^n = \left(1 - n\left(\frac{\pi^2}{8n^2} - O\left(\frac{1}{n^4}\right)\right)\right) \to 1$$

$$L = \lim_{n\to\infty}\left(\cos\frac{\pi}{2n}\right)^n \cdot \sin\left(\frac{\pi}{n} + \frac{\pi}{2}\right) = 1.1$$

$$\boxed{L = 1}$$

ROMANIAN MATHEMATICAL MAGAZINE

JP.615. *In ΔABC the following relationship holds:*

$$\frac{a^4+b^4}{{h_c}^2}+\frac{b^4+c^4}{{h_a}^2}+\frac{c^4+a^4}{{h_b}^2}\geq 96r^2$$

Proposed by Nguyen Hung Cuong – Romania

Solution by Omkumar Rao – India

The altitudes are:

$$h_a=\frac{2S}{a}\ ,h_b=\frac{2S}{b}\ ,h_c=\frac{2S}{c}$$

$$LHS=\frac{c^2(a^4+b^4)+a^2(b^4+c^4)+b^2(c^4+a^4)}{4S^2}$$

$$By\ AM-GM, x^4+y^4\geq 2x^2y^2\ \ for\ all\ real\ x,y.$$

$$LHS\geq\frac{6a^2b^2c^2}{4S^2}$$

$$By\ extended\ law\ of\ sines:\ abc=4RS$$

$$Euler's\ Inequality: R\geq 2r$$

$$LHS\geq 96r^2$$

Equality holds if and only if both the AM – GM Inequality and Euler's Inequality are tight, which occurs if and only if ΔABC is equilateral.

Hence Proved

JP.614. *In ΔABC the following relationship holds:*

$$\frac{a^4+b^4}{h_c}+\frac{b^4+c^4}{h_a}+\frac{c^4+a^4}{h_b}\geq 288r^3$$

Proposed by Nguyen Hung Cuong – Romania

Solution by Omkumar Rao – India

$$The\ altitudes\ are\ given\ by:\ h_c=\frac{2S}{c}\ ,h_a=\frac{2S}{a}\ ,h_b=\frac{2S}{b}$$

$$:S-Area\ of\ \Delta ABC\ ,s-semiperimeter\ r-inradius\ ,R-circumradius$$

$$LHS=\frac{c(a^4+b^4)}{2S}+\frac{a(b^4+c^4)}{2S}+\frac{b(c^4+a^4)}{2S}$$

$$By\ AM-GM, x^4+y^4\geq 2x^2y^2\ for\ all\ real\ x,y.$$

$$LHS\geq\frac{a.2b^2c^2+b.2c^2a^2+c.2a^2b^2}{2S}$$

$$LHS \geq \frac{2abc(ab+bc+ca)}{2S}. \text{ By extended law of sines,}$$

$$abc = 4RS, \qquad LHS \geq 4R(ab+bc+ca)$$

$$\text{Euler's Inequality} : R \geq 2r$$

$$ab+bc+ca = s^2 + 4Rr + r^2$$

$$s \geq 3\sqrt{3}r$$

$$LHS \geq 4.2r.36r^2 = 288r^3$$

Equality holds when both Euler's Inequality and the side − inradius inequality are tight ,which occurs if and only if ΔABC is equilateral.

Hence Proved

UP.607. Calculate the integral:

$$I = \int_0^1 \frac{x \ln x}{(x+1)(x^2+1)} dx$$

Proposed by Vasile Mircea Popa – Romania

Solution by Omkumar Rao – India

$$I = \frac{1}{2}\int_0^1 \left(\frac{-\ln x}{x+1} + \frac{\ln x}{x^2+1} + \frac{x \ln x}{x^2+1}\right) dx$$

$$A = \int_0^1 \frac{\ln x}{x+1} dx, \qquad B = \int_0^1 \frac{\ln x}{x^2+1} dx, \qquad C = \int_0^1 \frac{x \ln x}{x^2+1} dx$$

$$I = \frac{1}{2}(-A+B+C), \qquad A = \sum_{n=0}^{\infty} (-1)^n \int_0^1 x^n \ln x \, dx$$

$$A = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} = -\frac{1}{2}\zeta(2) = -\frac{\pi^2}{12}$$

$$B = \sum_{r=0}^{\infty} (-1)^r \int_0^1 x^{2r} \ln x \, dx = -\sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^2} = -G$$

where G denotes Catalan's constant.

$$C = \int_0^1 \frac{x \ln x}{x^2+1} dx \Rightarrow (x \to x^2), \qquad C = \frac{1}{4}\int_0^1 \frac{\ln x}{1+x} dx = \frac{1}{4}A = -\frac{\pi^2}{48}$$

$$I = \frac{1}{2}\left(\frac{\pi^2}{12} - G - \frac{\pi^2}{48}\right)$$

$$I = \frac{\pi^2}{32} - \frac{G}{2}$$

UP.598. *Find the following limit:*

$$L = \lim_{x\to 0}\left(\frac{1}{x}\cdot\lim_{n\to\infty}\sum_{k=1}^{n} 3^{k-1}\left(\sin\frac{x}{3^k}\right)^3\right),$$

$$a \in R$$

Proposed by Marian Ursarescu, Florica Anastase – Romania

Solution by Omkumar Rao – India

$$(\sin\theta)^3 = \frac{3\sin\theta - \sin 3\theta}{4}, \qquad :\theta = \frac{x}{3^k}$$

$$3^{k-1}\left(\sin\frac{x}{3^k}\right)^3 = \frac{1}{4}\left(3^k \sin\frac{x}{3^k} - 3^{k-1}\sin\frac{x}{3^{k-1}}\right)$$

$$Let\ f(k) = 3^k \sin\frac{x}{3^k}\ , k = 0, 1, 2, 3, \ldots.$$

$$\sum_{k=1}^{n} 3^{k-1}\left(\sin\frac{x}{3^k}\right)^3 = \frac{1}{4}\sum_{k=1}^{n}\left(f(k) - f(k-1)\right) = \frac{1}{4}\left(f(n) - f(0)\right)$$

$$= \frac{1}{4}\left(3^n \sin\frac{x}{3^n} - \sin x\right)$$

$$As\ n \to \infty, x.\frac{\sin\frac{x}{3^n}}{\frac{x}{3^n}} \to x, \qquad L = \lim_{x\to 0}\frac{1}{x}.\frac{x - \sin x}{4}$$

$$x - \sin x = \frac{x^3}{6} + O(x^5), \qquad \lim_{x\to 0}\frac{x - \sin x}{4x} = \frac{x^2}{24} + O(x^4) \to 0$$

$$as\ x \to 0$$

$$\boxed{L = 0}$$

JP.602. *If* $a, b, c > 0$ *and* $\lambda \geq 0$

then:

$$2\sum\frac{a^3}{b+\lambda c} + \frac{\lambda+1}{2}\sum bc \geq 2\sum a^2$$

Proposed by Marin Chirciu – Romania

Solution by Omkumar Rao – India

$$\sum_{cyc}\frac{a^3}{b+\lambda c} = \sum_{cyc}\frac{a^4}{ab+\lambda ac}$$

$$Applying\ (Cauchy - Schwarz, Engel\ form):$$

$$\sum_{cyc}\frac{a^4}{ab+\lambda ac}\geq\frac{(\sum a^2)^2}{\lambda\sum ab+\sum ab}$$

$$2\sum_{cyc}\frac{a^3}{b+\lambda c}\geq 2\frac{(\sum a^2)^2}{(\lambda+1)\sum ab}$$

$$2\sum_{cyc}\frac{a^3}{b+\lambda c}+\frac{\lambda+1}{2}\sum ab\geq 2\frac{(\sum a^2)^2}{(\lambda+1)\sum ab}+\frac{\lambda+1}{2}\sum ab$$

Applying AM − GM Inequality on RHS:

$$2\frac{(\sum a^2)^2}{(\lambda+1)\sum ab}+\frac{\lambda+1}{2}\sum ab\geq 2\sqrt{\frac{2(\sum a^2)^2}{(\lambda+1)\sum ab}\cdot\frac{\lambda+1}{2}\sum ab}$$

$$2\sum_{cyc}\frac{a^3}{b+\lambda c}+\frac{\lambda+1}{2}\sum ab\geq 2\sum a^2$$

$$Therefore, LHS\geq 2\sum a^2=RHS$$

Hence Proved.

UP.594. Solve the system:

$$x-2y+z+2=k^2,$$
$$with\ 3<k<11$$
$$x^2+y^2+z^2=109659$$
$$-x^4+y^2+z^2=80929$$
$$3<x<y<z,\qquad x,y,z\in N$$

Proposed by Said Attaoui − Algeria

Solution by Omkumar Rao − India

$$(x^2+y^2+z^2)-(-x^4+y^2+z^2)=28730$$
$$x^4+x^2-28730=0,\qquad x^2=169\Rightarrow x=13$$
$$(x\in N),\qquad y^2+z^2=109490$$
$$z=2y+k^2-15,\qquad y^2+(2y+k^2-15)^2=109490$$
$$5y^2+4y(k^2-15)+(k^2-15)^2=109490$$

For y to have integer solutions , Δ must be a perfect square:

$$\Delta=547450-(k^2-15)^2,\qquad \Delta\ is\ a\ perfect\ square\ for\ k=10$$
$$y^2+68y-20453=0\Rightarrow y=113z\Rightarrow 311$$
$$:3<x<y<z\Rightarrow 3<13<113<311$$
$$\boxed{(x,y,z)=(13,113,311)}$$

ROMANIAN MATHEMATICAL MAGAZINE

JP.600. *Calculate the limit of sequence* $(a_n)_{n\geq 1}$ *defined by the following relationship*:

$$a_n = \frac{1}{n}\int_0^{\frac{\pi}{2}} \ln\left(1 + e^{n\sin^{-1}x}\right) dx$$

Proposed by Vasile Mircea Popa – Romania

Solution by Omkumar Rao – India

$$a_n \overset{(\sin^{-1}x \to x)}{=} \frac{1}{n}\int_0^{\frac{\pi}{2}} \ln(1+e^{nx})\cos x\, dx, a_n = \frac{1}{n}\int_0^{\frac{\pi}{2}} (nx + \ln(1+e^{-nx}))\cos x\, dx$$

$$a_n = \frac{\pi}{2} - 1 + \frac{1}{n}\int_0^{\frac{\pi}{2}} \ln(1+e^{-nx})\cos x\, dx$$

$$\lim_{n\to\infty} \frac{1}{n}\int_0^{\frac{\pi}{2}} \ln(1+e^{-nx})\cos x\, dx$$

$$0 \leq \frac{1}{n}\int_0^{\frac{\pi}{2}} \ln(1+e^{-nx})\cos x\, dx \leq \frac{1}{n}\int_0^{\frac{\pi}{2}} \ln 2\, dx = \frac{\pi \ln 2}{2n}$$

As $n \to \infty, \frac{\pi \ln 2}{2n} \to 0$. *By Squeeze theorem* :

$$\frac{1}{n}\int_0^{\frac{\pi}{2}} \ln(1+e^{-nx})\cos x\, dx = 0$$

$$\boxed{\lim_{n\to\infty} a_n = \frac{\pi}{2} - 1}$$

UP.599. *Calculate the integral*:

$$\int_{-\pi}^{\pi} \frac{\cot^{-1}x}{\sqrt{3-\cos x}} dx$$

In this problem we will consider the definition of the function $\cot^{-1}x$ *which has the image the interval* $(0, \pi)$.

Proposed by Vasile Mircea Popa – Romania

Solution by Omkumar Rao – India

$$I=\int_{-\pi}^{\pi}\frac{\cot^{-1}x}{\sqrt{3-\cos x}}dx\Longrightarrow(1)$$

$$I\overset{(x\to -x)}{=}\int_{-\pi}^{\pi}\frac{\pi-\cot^{-1}x}{\sqrt{3-\cos x}}dx\Longrightarrow(2)$$

$$2I=\int_{-\pi}^{\pi}\frac{\pi}{\sqrt{3-\cos x}}dx,\qquad I=\pi\int_{0}^{\pi}\frac{1}{\sqrt{3-\cos x}}dx,\qquad I=\frac{\pi}{\sqrt{2}}\int_{0}^{\pi}\frac{1}{\sqrt{1+\left(\sin\frac{x}{2}\right)^2}}dx$$

$$I\overset{\left(\frac{x}{2}\to x\right)}{=}\sqrt{2}\pi\int_{0}^{\frac{\pi}{2}}\frac{dx}{\sqrt{1+(\sin x)^2}},I=\sqrt{2}\pi\int_{0}^{\frac{\pi}{2}}\frac{dx}{\sqrt{1+(\cos x)^2}},I=\sqrt{2}\pi\int_{0}^{\frac{\pi}{2}}\frac{dx}{\sqrt{2-(\sin x)^2}}$$

$$I=\pi\int_{0}^{\frac{\pi}{2}}\frac{dx}{\sqrt{1-\frac{(\sin x)^2}{2}}},\qquad I\overset{(\sin x\to x)}{=}\sqrt{2}\pi\int_{0}^{1}\frac{dx}{\sqrt{1-x^2}\sqrt{2-x^2}}$$

$$I\overset{(x^2\to x)}{=}\frac{\pi}{\sqrt{2}}\int_{0}^{1}\frac{dx}{\sqrt{x}\sqrt{1-x}\sqrt{2-x}},\qquad I\overset{(1-x\to x)}{=}\frac{\pi}{\sqrt{2}}\int_{0}^{1}\frac{dx}{\sqrt{x}\sqrt{1-x^2}}$$

$$I\overset{(x^2\to x)}{=}\frac{\pi}{2\sqrt{2}}\int_{0}^{1}x^{\frac{-3}{4}}(1-x)^{\frac{-1}{2}}dx,\qquad I=\frac{\pi}{2\sqrt{2}}\int_{0}^{1}x^{\frac{1}{4}-1}(1-x)^{\frac{1}{2}-1}dx$$

$$I=\frac{\pi}{2\sqrt{2}}\beta\left(\frac{1}{4},\frac{1}{2}\right),\qquad I=\frac{\pi}{2\sqrt{2}}\frac{\Gamma\left(\frac{1}{4}\right).\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)},\qquad I=\frac{\pi}{2\sqrt{2}}\frac{\Gamma\left(\frac{1}{4}\right).\sqrt{\pi}}{\left(\frac{\pi\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)}\right)}$$

$$\boxed{\int_{-\pi}^{\pi}\frac{\cot^{-1}x}{\sqrt{3-\cos x}}dx=\frac{\sqrt{\pi}}{4}\left(\Gamma\left(\frac{1}{4}\right)\right)^2}$$

UP.612. If $f:R^*_+\to R^*_+$ is a continuous function and $(a_n)_{n\geq 1}$ is defined by $a_1=a_2=1$,

$$a_{n+1}=\sum_{k=1}^{n}\frac{a_k}{k},\forall n\geq 2\ and\ (x_n)_{n\geq 1},x_n=\sum_{k=1}^{n}\frac{1}{a_k},find$$

$$\lim_{n\to\infty}\frac{1}{n}\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx$$

Proposed by D.M. Batinetu – Giurgiu, Neculai Stanciu – Romania

Solution by Omkumar Rao – India

$$I_n=\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx\Rightarrow(1)$$

$$(x\to e^{x_n}+e^{x_{n+1}}-x):$$

$$I_n=\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(e^{x_{n+1}}-x)}{f(x-e^{x_n})+f(e^{x_{n+1}}-x)}dx\Rightarrow(2)$$

$$2I_n=\int_{e^{x_n}}^{e^{x_{n+1}}}1\,dx$$

$$I_n=\frac{(e^{x_{n+1}}-e^{x_n})}{2},L=\lim_{n\to\infty}\frac{1}{2n}(e^{x_{n+1}}-e^{x_n})$$

$$a_{n+1}=\sum_{k=1}^{n}\frac{a_k}{k}=\left(\sum_{k=1}^{n-1}\frac{a_k}{k}\right)+\frac{a_n}{n}=a_n+\frac{a_n}{n},\qquad a_n=\frac{n}{2}$$

$$x_n=\sum_{k=1}^{n}\frac{1}{a_k}=\frac{1}{a_1}+\sum_{k=2}^{n}\frac{2}{k},\qquad x_n=1+2(H_n-1)=2H_n-1$$

$$H_n=\gamma+\ln n+\frac{1}{2n}+O\left(\frac{1}{n^2}\right),\qquad x_n=2\gamma+2\ln n+\frac{1}{n}+2O\left(\frac{1}{n^2}\right)-1$$

$$e^{x_n}=e^{(2\gamma-1)+2O\left(\frac{1}{n^2}\right)+\ln n^2}=n^2e^{2\gamma-1}e^{2O\left(\frac{1}{n^2}\right)},e^{2O\left(\frac{1}{n^2}\right)}=1+2O\left(\frac{1}{n^2}\right)\approx1$$

$$e^{x_n}=n^2e^{2\gamma-1},\qquad x_{n+1}=x_n+\frac{1}{a_{n+1}}$$

$$L=\lim_{n\to\infty}\frac{e^{x_n}\left(e^{\frac{2}{n+1}}-1\right)}{2n},\qquad L=\lim_{n\to\infty}\frac{n^2e^{2\gamma-1}(2)}{2n(n+1)}$$

$$\boxed{L=e^{2\gamma-1}}$$

UP.610. If $f:R_+^*\to R_+^*$ is continuous function and

$$\gamma_n=-\ln n+\sum_{k=1}^{n}\frac{1}{k},find\ \lim_{n\to\infty}n\int_{\gamma}^{\gamma_n}\frac{f(x-\gamma)}{f(\gamma_n-x)+f(x-\gamma)}dx$$

Proposed by D.M.Batinetu – Giurgiu, Neculai Stanciu – Romania

Solution by Omkumar Rao – India

$$I_n = \int_{\gamma}^{\gamma_n} \frac{f(x-\gamma)}{f(\gamma_n - x) + f(x-\gamma)}\,dx \Rightarrow (1)$$

$$(x \longrightarrow \gamma + \gamma_n - x):$$

$$I_n = \int_{\gamma}^{\gamma_n} \frac{f(\gamma_n - x)}{f(x-\gamma) + f(\gamma_n - x)}\,dx \Rightarrow (2)$$

$$2I_n = \int_{\gamma}^{\gamma_n} 1\,dx, \qquad I_n = \frac{(\gamma_n - \gamma)}{2}$$

$$L = \lim_{n\to\infty} n\frac{(\gamma_n - \gamma)}{2}, \qquad \gamma_n = -\ln n + \sum_{k=1}^{n}\frac{1}{k}$$

$$\gamma_n - \gamma = -\ln n + \sum_{k=1}^{n}\frac{1}{k} - \gamma$$

$$\sum_{k=1}^{n}\frac{1}{k} : H_n \; as \; n \to \infty$$

$$H_n = \ln n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)$$

$$\lim_{n\to\infty} n\frac{-\gamma - \ln n + \ln n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)}{2} = \lim_{n\to\infty} n\frac{\frac{1}{2n} + O\left(\frac{1}{n^2}\right)}{2}$$

$$\boxed{L = \frac{1}{4}}$$

JP.613. Find the angle between the lines:

$$d_1: \begin{cases} x = 3 - 2t \\ y = 1 + t \\ z = -1 + 3t \end{cases} ; \; d_2: \begin{cases} x = 2 + 4t \\ y = -1 + 2t \\ z = 4 - t \end{cases}$$

$$(t \in R)$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

The angle between two lines is determined by the angle between their direction vectors. For line d_1, the direction vector is $\vec{v}_1 = (-2, 1, 3)$, and for line d_2, the direction vector: $\vec{v}_2 = (4, 2, -1)$

The angle θ between the lines is given by:

$$\cos\theta = \frac{|\vec{v}_1 . \vec{v}_2|}{|\vec{v}_1| . |\vec{v}_2|}, \qquad \vec{v}_1 . \vec{v}_2 = -9, |\vec{v}_1| = \sqrt{14}\,, |\vec{v}_2| = \sqrt{21}$$

$$\cos\theta = \frac{3\sqrt{6}}{14}$$

The angle between the lines d_1 and d_2 is

$$\theta = \cos^{-1}\left(\frac{3\sqrt{6}}{14}\right)$$

JP.612. *Find the angle between the line d and the real plane P.*

$$d:\begin{cases} x = 2 + 3t \\ y = 3 - 2t \\ z = 1 + 5t \end{cases}$$

$$t \in R \ ; \quad P: 2x + y + z - 5 = 0$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

The angle between a line and a plane is determined by the angle between the line's direction vector and the plane's normal sector. For line d, the direction vector is $\vec{d} = (3, -2, 5)$, and for plane P, the normal vector is $\vec{n} = (2, 1, 1)$. The angle θ between the line and the plane is given by:

$$\sin\theta = \frac{|\vec{d}.\vec{n}|}{|\vec{d}|.|\vec{n}|}, \vec{d}.\vec{n} = 9$$

$$|\vec{d}| = \sqrt{38}\ , \qquad |\vec{n}| = \sqrt{6}$$

$$\sin\theta = \frac{3\sqrt{57}}{38}$$

The angle between the line d and the plane P is $\theta = \sin^{-1}\left(\frac{3\sqrt{57}}{38}\right)$

SP.593. *Solve for real numbers:*

$$(\sin x + \cos y)^2 = (\sin x + 1)(\cos y - 1)$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$$m = \sin x \in [-1, 1], n = \cos y \in [-1, 1]:$$

$$(m + n)^2 = (m + 1)(n - 1), m^2 + m(n + 1) + (n^2 - n + 1) = 0$$

$$\Delta = (n + 1)^2 - 4(n^2 - n + 1)$$

$$\Delta = -3(n - 1)^2$$

$$\Delta \geq 0 \Rightarrow (n - 1)^2 \leq 0 \Rightarrow n = 1, \Delta = 0 \Rightarrow m^2 + 2m + 1 = 0 \Rightarrow m = -1$$

$$\cos y = 1 \Rightarrow y = 2r\pi,$$

$$\sin x = -1 \Rightarrow x = -\frac{\pi}{2} + 2p\pi,$$

$r, p \in Z.$

$$\boxed{(x, y) = \left(-\frac{\pi}{2} + 2p\pi, 2r\pi\right), r, p \in Z}$$

JP.588. *If* $a, b, c, d > 0, abcd = 1$ *then*:

$$a^{b+c+d} b^{c+d+a} c^{a+b+d} d^{a+b+c} \leq 1$$

Proposed by Marin Chirciu – Romania

Solution by Omkumar Rao – India

$$\text{Let } m = a + b + c + d, \qquad a^{m-a} b^{m-b} c^{m-c} d^{m-d} \leq 1$$

$$(abcd)^m \leq a^a b^b c^c d^d, a^a b^b c^c d^d \geq 1$$

$$a \ln a + b \ln b + c \ln c + d \ln d \geq 0$$

$$\text{Let } f(x) = x \ln x, \qquad f'^{(x)} = 1 + \ln x, f'' = \frac{1}{x} > 0 \text{ for } x > 0$$

Hence f is a strictly convex function on $(0, \infty)$.

Apply Jensen's Inequality:

$$\frac{f(a) + f(b) + f(c) + f(d)}{4} \geq f\left(\frac{a + b + c + d}{4}\right)$$

$$\frac{a \ln a + b \ln b + c \ln c + d \ln d}{4} \geq \left(\frac{a + b + c + d}{4}\right) \ln\left(\frac{a + b + c + d}{4}\right)$$

$$\text{By AM} - \text{GM Inequality}: \frac{a + b + c + d}{4} \geq \sqrt[4]{abcd} = \sqrt[4]{1} = 1$$

$$\ln\left(\frac{a + b + c + d}{4}\right) \geq 0$$

$$\frac{a \ln a + b \ln b + c \ln c + d \ln d}{4} \geq (1)(0)$$

$$a \ln a + b \ln b + c \ln c + d \ln d \geq 0$$

This proves that $a^a b^b c^c d^d \geq 1$, *and hence*

$$\boxed{a^{b+c+d} b^{c+d+a} c^{a+b+d} d^{a+b+c} \leq 1}$$

Hence Proved

SP.595. *Solve for real numbers*:

$$\tan 2x + \tan 3x + \tan 5x = \tan 2x \cdot \tan 3x \cdot \tan 5x$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$$\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-(\tan A\tan B+\tan B\tan C+\tan C\tan A)}$$

$$\tan 2x+\tan 3x+\tan 5x=\tan 2x\,.\tan 3x\,.\tan 5x$$

$$\tan(A+B+C)=0\Leftrightarrow A+B+C=r\pi\,,r\in Z.$$

$$10x=r\pi$$

$$x=\frac{r\pi}{10}$$

$$5x=\frac{r\pi}{2}\notin\frac{\pi}{2}+r\pi\Rightarrow r\ even, r=2k$$

$$\boxed{x=\frac{k\pi}{5}\,,k\in Z}$$

UP.591. Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{3^n}\sum_{k=0}^{n-1}\sqrt{\binom{n}{k}\binom{n}{k+1}}$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$$\sqrt{\binom{n}{k}\binom{n}{k+1}}\le\frac{\binom{n}{k}+\binom{n}{k+1}}{2},\qquad\sum_{k=0}^{n-1}\sqrt{\binom{n}{k}\binom{n}{k+1}}\le\frac{1}{2}\left(\sum_{k=0}^{n-1}\binom{n}{k}+\sum_{k=0}^{n-1}\binom{n}{k+1}\right)$$

$$\sum_{k=0}^{n-1}\sqrt{\binom{n}{k}\binom{n}{k+1}}\le\frac{1}{2}\left(2(2^n-1)\right),\qquad 0\le\sum_{k=0}^{n-1}\sqrt{\binom{n}{k}\binom{n}{k+1}}\le(2^n-1)$$

$$0\le\lim_{n\to\infty}\frac{1}{3^n}\sum_{k=0}^{n-1}\sqrt{\binom{n}{k}\binom{n}{k+1}}\le\lim_{n\to\infty}\frac{(2^n-1)}{3^n}$$

$$\lim_{n\to\infty}\frac{(2^n-1)}{3^n}=0$$

By Squeeze theorem:

$$\boxed{\lim_{n\to\infty}\frac{1}{3^n}\sum_{k=0}^{n-1}\sqrt{\binom{n}{k}\binom{n}{k+1}}=0}$$

SP.615. Let be $A(1,2,3)$ and the real plane: $P: 2x+y+z-5=0$. Find the parametrical equations of the perpendicular line from A to the real plane P.

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

A line $\perp$ to a plane is parallel to the normal vector of that plane.

$$\text{Direction vector of the line is } \vec{n} = \langle a, b, c\rangle = (2,1,1)$$
$$A(X,Y,Z) = (1,2,3), x = X + a\lambda, y = Y + b\lambda, z = Z + c\lambda$$
$$\begin{cases} x = 1 + 2\lambda \\ y = 2 + \lambda \\ z = 3 + \lambda \end{cases}$$
$$\lambda \in R$$

SP. 614. *Let be $A(1,3,2); B(3,1,1); C(4,2,0)$.*
Find the area of the parallelogram builded on $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

Let the given vertices of the triangle in the three – dimensional Cartesian coordinate system R^3 be denoted by their position vectors with respect to the origin: $r_A = (1,3,2), r_B = (3,1,1), r_C = (4,2,0)$
A parallelogram embedded in R^3 defined by two adjacent sides can be characterized by the directed line segments representing those sides.
$$\overrightarrow{AB} = r_B - r_A = 2\hat{\imath} - 2\hat{\jmath} - \hat{k}$$
$$\overrightarrow{AC} = r_C - r_A = 3\hat{\imath} - \hat{\jmath} - 2\hat{k}$$
where $\hat{\imath}, \hat{\jmath}, \hat{k}$ constitute the standard orthonormal basis for R^3.
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -2 & -1 \\ 3 & -1 & -2 \end{vmatrix}, \overrightarrow{AB} \times \overrightarrow{AC} = (3,1,4)$$
$$\text{Area} = \left|\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|\right| = \sqrt{26} \text{ square units}$$

UP. 592. *Solve for real numbers:*
$$\begin{cases} \cos x + \cos y + \cos z = 1 \\ (\cos x)^2 + (\cos y)^2 + (\cos z)^2 = 1 \\ (\cos x)^3 + (\cos y)^3 + (\cos z)^3 = 1 \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$$m = \cos x\ , n = \cos y\ , p = \cos z$$
$$m, n, p \in [-1, 1]$$

$$\begin{cases} S_1 = a+b+c \\ S_2 = ab+bc+ca \\ S_3 = abc \end{cases}$$
$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$$
$$1^2 = 1+2(ab+bc+ca)$$
$$S_2 = ab+bc+ca = 0$$
$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$
$$1-3S_3 = (1)(1-0)$$
$$S_3 = 0$$
Applying Vieta's relations:
$$m^3 - S_1 m^2 + S_2 m - S_3 = 0$$
$$m^3 - m^2 = 0 \Rightarrow m = 0,0,1$$

$$(\cos x, \cos y, \cos z) = (1,0,0)$$
$$(\cos x, \cos y, \cos z) = (0,1,0)$$
$$(\cos x, \cos y, \cos z) = (0,0,1)$$
$$\cos Y = 1 \Rightarrow Y = 2r\pi \text{ and}$$
$$\cos Y = 0 \Rightarrow Y = \frac{\pi}{2} + j\pi$$
Solution set consisting of all three permutations:
$$\left(2r\pi, \frac{\pi}{2}+j\pi, \frac{\pi}{2}+p\pi\right)$$
where $r, j, p \in Z$

UP.586. *If* $M_{2,1}(R); B \in M_{1,2}(R);$ $AB = \begin{pmatrix} 0 & 0 \\ 8 & 1 \end{pmatrix}$ *then find* BA.

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$$A = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad B = (c \quad d)$$
$$A.B = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}$$
$$ac = 0, ad = 0, bc = 8, bd = 1$$
$$B.A = (ca+bd)$$
$$\boxed{B.A = (1)}$$

JP.603. *If* $a, b, c > 0$ *then*:

$$2\left(a^a + b^b + c^c\right) \geq a^b + a^c + b^a + b^c + c^a + c^b$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$$\text{for } a \geq b, a^a + b^b \geq a^b + b^a \Rightarrow (1)$$
$$\text{for } b \geq c, b^b + c^c \geq b^c + c^b \Rightarrow (2)$$
$$\text{for } c \geq a, c^c + a^a \geq c^a + a^c \Rightarrow (3)$$

Adding all three equations gives:

$$2\left(a^a+b^b+c^c\right)\geq a^b+a^c+b^a+b^c+c^a+c^b$$

Hence Proved

JP.601. *If* $a,b,c>0$ *and* $\lambda\geq 0$ *then*:

$$(\lambda+1)\sum\frac{a^2}{b+\lambda c}+2\sqrt{3(ab+bc+ca)}\geq 3(a+b+c)$$

Proposed by Marin Chirciu – Romania

Solution by Omkumar Rao – India

$$\text{Cauchy}-\text{Schwarz Inequality}:\ \sum_{cyc}\frac{a^2}{b+\lambda c}\geq\frac{(a+b+c)^2}{(\lambda+1)(a+b+c)}$$

$$(\lambda+1)\sum_{cyc}\frac{a^2}{b+\lambda c}\geq a+b+c$$

$$LHS\geq(a+b+c)+2\sqrt{3(ab+bc+ca)}$$

$$(a+b+c)+2\sqrt{3(ab+bc+ca)}\geq 3(a+b+c)$$

$$3(ab+bc+ca)\geq(a+b+c)^2,\qquad a^2+b^2+c^2-ab-bc-ca\geq 0$$

$$\frac{1}{2}((a-b)^2+(b-c)^2+(c-a)^2)\geq 0$$

Since squares of real numbers are always non – negative, this inequality is true.
Hence Proved

UP.613. *Let be* $A(1,2,0);B(2,0,1);C(3,3,3)$. *Find the area of* ΔABC.

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$$\overrightarrow{AB}=(1,-2,1),\overrightarrow{AC}=(2,1,3),\qquad \overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\1&-2&1\\2&1&3\end{vmatrix}$$

$$=\hat{\imath}\left((-2)(3)-(1)(1)\right)-\hat{\jmath}\left((1)(3)-(1)(2)\right)+\hat{k}\left((1)(1)-(-2)(2)\right)$$

$$=(-7,-1,5),\quad\left|\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|\right|=5\sqrt{3}$$

$$\text{Area}=\frac{1}{2}\left|\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|\right|=\frac{5\sqrt{3}}{2}$$

SP.612. *Let be* $A(1,2,3); B(4,1,1); C(0,3,2); D(2,1,3)$. *Find the volume of the parallelipiped builded on the vectors* $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$.

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$$\overrightarrow{AB} = (3,-1,-2),$$
$$\overrightarrow{AC} = (-1,1,-1), \overrightarrow{AD} = (1,-1,0)$$
$$Volume = \left|\left(det(V)\right)\right| = \overrightarrow{AB}.\left(\overrightarrow{AC} \times \overrightarrow{AD}\right)$$
$$det(V) = \begin{vmatrix} 3 & -1 & -2 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$$
$$det(V) = -2$$

The volume of the parallelipiped is 2 cubic units.

SP.613. *Let be* $A(2,4,1); B(4,2,5)$.
Find the mediator plane of the segment $[AB]$.

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

The mediator plane passes through the midpoint C of the segment connecting A and B.

$$C = \left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}, \frac{z_A + z_B}{2}\right), \qquad C = \left(\frac{2+4}{2}, \frac{4+2}{2}, \frac{1+5}{2}\right) = (3,3,3)$$

The plane is $\perp$ *to the segment* $[AB]$.

Therefore, the vector $\overrightarrow{AB}$ *serves as a normal vector* $\vec{n}$ *to the plane*:

$$\vec{n} = (x_B - x_A)\hat{\imath} + (y_B - y_A)\hat{\jmath} + (z_B - z_A)\hat{k}$$
$$\vec{n} = (2,-2,4) \Rightarrow \vec{n} = (1,-1-2)$$

General equation of a plane with normal vector (a,b,c) *passing through a point* (X,Y,Z) *is*:

$$a(x-X) + b(y-Y) + c(z-Z) = 0$$
$$1(x-3) - 1(y-3) + 2(z-3) = 0$$
$$x - y + 2z - 6 = 0$$

:The equation of the mediator plane is $x - y + 2z - 6 = 0$

ROMANIAN MATHEMATICAL MAGAZINE

JP.594. *Solve for real numbers*:

$$(\sin x)^{2022}(\cos x)^{2024} = \frac{1}{2^{2022}}$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$$(\sin x)^{2022}(\cos x)^{2024} = (\sin x \cos x)^{222}(\cos x)^2$$

$$\left(\frac{\sin 2x}{2}\right)^{2022}(\cos x)^2 = \frac{1}{2^{2022}},\quad (\sin 2x)^{2022}(\cos x)^2 = 1$$

$$|\sin 2x| \le 1, (\cos x)^2 \le 1$$

$$\textit{Equality needs } (\sin 2x)^{2022} = 1 \textit{ and } (\cos x)^2 = 1$$

$$(\cos x)^2 = 1 \Rightarrow x = r\pi$$

$$2x = 2\,r\pi \Rightarrow \sin 2x = 0$$

$$(\sin 2x)^{2022} = 0 \not\Rightarrow 1$$

$$\boxed{:x \in \emptyset}$$

SP.590. *Solve the following system in integers* $(x, y, z) \in N^* \times N^* \times Z$.

$$\begin{cases} x^3 - y^2 + 2z = 0 \\ x^2 + y^2 + z^2 = 179 \end{cases}$$

Proposed by Said Attaoui – Algeria

Solution by Omkumar Rao – India

$$\textit{Adding both equations}: x^3 + x^2 + z^2 + 2z = 179 \Rightarrow x^3 + x^2 + (z+1)^2 = 180$$

$$(z+1)^2 = 180 - x^3 - x^2 \Rightarrow x^3 + x^2 \le 180 \Rightarrow x \in \{1,2,3,4,5\}\ \textit{for } x \in N^*$$

$$\Rightarrow x \in \{1,2,4,5\}\ \textit{rejects as } 180 - x^3 - x^2 \textit{ is not a perfect square.}$$

$$\textit{For } x = 3 \Rightarrow z \in \{11, -13\}$$

$$:\begin{cases} z = 11 \Rightarrow y = 7 \\ z = -13 \Rightarrow y = 1 \end{cases}$$

$$\boxed{(x, y, z) \in (3,7,11), (3,1,-13)}$$

SP.586. ***Solve for real numbers:***

$$\begin{cases} \left(\sqrt{x+y}-\sqrt{x}\right)\left(\sqrt{x^2+xy}+1\right)=xy \\ x+y+z=3 \\ \left(\sqrt{y+z}-\sqrt{y}\right)\left(\sqrt{y^2+yz}+1\right)=zy \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$$\frac{y\left(\sqrt{x}\sqrt{x+y}+1\right)}{\left(\sqrt{x+y}+\sqrt{x}\right)}=xy \overset{t=\sqrt{x}}{\Rightarrow} t\sqrt{t^2+y}+1=t^2\sqrt{t^2+y}+t^3$$

$$1-t^3=\sqrt{t^2+y}.(t^2-t)\Rightarrow(1-t)\left(1+t+t^2+t\sqrt{t^2+y}\right)=0$$

$$(1-t)\left(1+t+t^2+t\sqrt{t^2+y}\right)=0\Rightarrow t=1\Rightarrow x=1$$

$$\frac{z\left(\sqrt{y}\sqrt{y+z}+1\right)}{\left(\sqrt{y+z}+\sqrt{y}\right)}=yz \overset{k=\sqrt{y}}{\Rightarrow} k\sqrt{k^2+z}+1=k^2\sqrt{k^2+z}+k^3$$

$$1-k^3=\sqrt{k^2+z}.(k^2-k)\Rightarrow(1-k)\left(1+k+k^2+k\sqrt{k^2+z}\right)=0$$

$$(1-k)\left(1+k+k^2+k\sqrt{k^2+z}\right)=0\Rightarrow k=1\Rightarrow y=1$$

$$x+y+z=3\Rightarrow z=1$$

Thus, the solution set is $(x,y,z)=(1,1,1)$

JP.592. ***Solve for real numbers:***

$$2^x+3^y+5^z=10$$

$$\left|\sqrt{x^2+y^2+z^2}-\sqrt{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}}\right|=\left|x-\frac{1}{x}\right|+\left|y-\frac{1}{y}\right|+\left|z-\frac{1}{z}\right|$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$$Let\ \vec{a}=(x,y,z)\ and\ \vec{b}=\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$$

$$\textbf{By Triangle Inequality}: \ \big||a|-|b|\big| \le |a-b|$$

$$\left|\sqrt{x^2+y^2+z^2}-\sqrt{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}}\right| \le \sqrt{\left(x-\frac{1}{x}\right)^2+\left(y-\frac{1}{y}\right)^2+\left(z-\frac{1}{z}\right)^2}$$

$$\textbf{To show}: \ \sqrt{\left(x-\frac{1}{x}\right)^2+\left(y-\frac{1}{y}\right)^2+\left(z-\frac{1}{z}\right)^2} \le \left|x-\frac{1}{x}\right|+\left|y-\frac{1}{y}\right|+\left|z-\frac{1}{z}\right|$$

$$\textbf{Let } m=\left|x-\frac{1}{x}\right| , n=\left|y-\frac{1}{y}\right| , p=\left|z-\frac{1}{z}\right| , m,n,p>0$$

$$m^2+n^2+p^2 \le (m+n+p)^2 \Rightarrow 2(mn+np+pm) \ge 0$$

$$\left|\sqrt{x^2+y^2+z^2}-\sqrt{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}}\right| \le \left|x-\frac{1}{x}\right|+\left|y-\frac{1}{y}\right|+\left|z-\frac{1}{z}\right|$$

$$\textbf{According to question}:$$

$$\left|\sqrt{x^2+y^2+z^2}-\sqrt{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}}\right| = \left|x-\frac{1}{x}\right|+\left|y-\frac{1}{y}\right|+\left|z-\frac{1}{z}\right|$$

$$\frac{x}{\left(\frac{1}{x}\right)}=\frac{y}{\left(\frac{1}{y}\right)}=\frac{z}{\left(\frac{1}{z}\right)}=k \Rightarrow x^2=y^2=z^2=\frac{1}{k}$$

$$x-\frac{1}{x}=y-\frac{1}{y}=z-\frac{1}{z}=0 \Rightarrow x,y,z \in \{-1,1\}$$

$$\boxed{\textbf{Thus, the unique solution is } (x,y,z)=(1,1,1)}$$

JP.587. *If $x,y \ge 1$ then:*

$$\ln xy\,(2\ln xy+1) \ge 4\left(\ln x\sqrt{\ln y}+\ln y\sqrt{\ln x}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$$\textbf{Let } m=\sqrt{\ln x} \textbf{ and } n=\sqrt{\ln y}$$

$$(m^2+n^2)(2m^2+2n^2+1) \ge 4(m^2n+n^2m)$$

$$4mn(m+n) \le (m+n)^2(m+n) = (m+n)^3$$

$$(m^2+n^2)(2m^2+2n^2+1) \ge (m+n)^3$$

$$(m+n)^3 = (m+n)^2(m+n) \le 2(m^2+n^2)(m+n)$$

$$2(m^2+n^2)(m+n) \le (m^2+n^2)(2m^2+2n^2+1)$$

$$2(m+n) \le (2m^2+2n^2+1)$$

$$m^2-2m+2n^2-2n+1 \ge 0$$

$$\Rightarrow \left(m-\frac{1}{2}\right)^2+\left(n-\frac{1}{2}\right)^2+m^2+n^2 \ge 0$$
$$which\ holds\ for\ all\ m,n \ge 0$$

UP.608. *If* $f:R \to (1,\infty)$ *and* $f:R \to R$ *are continuous functions, and* $(L_n)_{n\ge 0}$ *is Lucas sequence, and* $y_n = \sqrt[n]{(2n-1)!!\,L_n}$, $n \in N^* - \{1\}$, *find:*

$$\lim_{n\to\infty} \int_{y_n}^{y_{n+1}} \frac{\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}{\left(f(x-y_n)\right)^{g(y_{n+1}-x)}+\left(f(y_{n+1}-x)\right)^{g(x-y_n)}}\,dx$$

Proposed by D. M. Batinetu – Giurgiu, Neculai Stanciu – Romania

Solution by Omkumar Rao – India

$$I_n = \int_{y_n}^{y_{n+1}} \frac{\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}{\left(f(x-y_n)\right)^{g(y_{n+1}-x)}+\left(f(y_{n+1}-x)\right)^{g(x-y_n)}}\,dx \Rightarrow (1)$$

$$I_n \overset{x\to y_{n+1}+y_n-x}{\Longrightarrow} \int_{y_n}^{y_{n+1}} \frac{\left(f(y_{n+1}-x)\right)^{g(x-y_n)}}{\left(f(x-y_n)\right)^{g(y_{n+1}-x)}+\left(f(y_{n+1}-x)\right)^{g(x-y_n)}}\,dx \Rightarrow (2)$$

$$I_n = \frac{(y_{n+1}-y_n)}{2}$$

$$Let\ K_n = (2n-1)!!\,L_n \overset{D'Alembert's\ Theorem}{\Longrightarrow} \lim_{n\to\infty}\frac{y_n}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{K_n}{n^n}}$$

$$\lim_{n\to\infty}(y_{n+1}-y_n)=\lim_{n\to\infty}\frac{y_n}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{K_n}{n^n}}$$

$$\lim_{n\to\infty}\frac{K_{n+1}}{K_n}\frac{n^n}{(n+1)^{n+1}}=\frac{1}{e}\lim_{n\to\infty}\frac{L_{n+1}}{L_n}\frac{(2n+1)!!}{n(2n-1)!!}=\frac{2}{e}\lim_{n\to\infty}\frac{L_{n+1}}{L_n}$$

$$Let\ \lim_{n\to\infty}\frac{L_{n+1}}{L_n}=k$$

$$L_{n+1}=L_n+L_{n-1}\Rightarrow\frac{L_{n+1}}{L_n}=1+\frac{L_{n-1}}{L_n}$$

$$\lim_{n\to\infty}\frac{L_{n+1}}{L_n}=\lim_{n\to\infty}1+\lim_{n\to\infty}\frac{L_{n-1}}{L_n}$$

$$k=1+\frac{1}{k}\Rightarrow k^2-k-1=0\Rightarrow k=\phi$$

$$\boxed{\lim_{n\to\infty}\frac{(y_{n+1}-y_n)}{2}=\frac{\phi}{e}=\frac{1+\sqrt{5}}{2e}}$$

UP.606. *If $f:R\longrightarrow(1,\infty)$ and $g:R\to R$ are continuous functions, and $y_n=\sqrt[n]{n!\,F_n}$, $n\in N^*-\{1\}$, where $(F_n)_{n\geq 0}$ is Fibonacci's sequence, find:*

$$\lim_{n\to\infty}\int_{y_n}^{y_{n+1}}\frac{\big(f(x-y_n)\big)^{g(y_{n+1}-x)}}{\big(f(x-y_n)\big)^{g(y_{n+1}-x)}+\big(f(y_{n+1}-x)\big)^{g(x-y_n)}}dx$$

Proposed by D.M.Batinetu – Giurgiu, Neculai Stanciu – Romania

Solution by Omkumar Rao – India

$$I_n=\int_{y_n}^{y_{n+1}}\frac{\big(f(x-y_n)\big)^{g(y_{n+1}-x)}}{\big(f(x-y_n)\big)^{g(y_{n+1}-x)}+\big(f(y_{n+1}-x)\big)^{g(x-y_n)}}dx\Rightarrow(1)$$

$$I_n\overset{x\to y_{n+1}+y_n-x}{\rightleftharpoons}\int_{y_n}^{y_{n+1}}\frac{\big(f(y_{n+1}-x)\big)^{g(x-y_n)}}{\big(f(x-y_n)\big)^{g(y_{n+1}-x)}+\big(f(y_{n+1}-x)\big)^{g(x-y_n)}}dx\Rightarrow(2)$$

$$I_n = \frac{(y_{n+1} - y_n)}{2}$$

$$Let\ K_n = n!\,F_n \overset{D'Alembert's\ Theorem}{\Longrightarrow} \lim_{n\to\infty} \frac{y_n}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{K_n}{n^n}}$$

$$\lim_{n\to\infty} (y_{n+1} - y_n) = \lim_{n\to\infty} \frac{y_n}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{K_n}{n^n}}$$

$$\lim_{n\to\infty} \frac{K_{n+1}}{K_n} \frac{n^n}{(n+1)^{n+1}} = \frac{1}{e} \lim_{n\to\infty} \frac{(n+1)!\,F_{n+1}}{n.n!\,F_n} = \frac{1}{e} \lim_{n\to\infty} \left(\frac{F_{n+1}}{F_n}\right) = \frac{\phi}{e}$$

$$\boxed{\lim_{n\to\infty} \frac{(y_{n+1} - y_n)}{2} = \frac{1+\sqrt{5}}{4e}}$$

SP. 605. Let $A(1,2,1); B(2,1,3); C(4,4,4)$. Find the distance from the point A to the line BC.

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$$\overrightarrow{BC} = \vec{C} - \vec{B} = (2,3,1) \quad , \quad \overrightarrow{BA} = \vec{A} - \vec{B} = (-1,1,-2), D = \frac{\left|\overrightarrow{BA} \times \overrightarrow{BC}\right|}{\left|\overrightarrow{BC}\right|}$$

$$\overrightarrow{BA} \times \overrightarrow{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = (7,-3,-5)$$

$$\left|\overrightarrow{BA} \times \overrightarrow{BC}\right| = \sqrt{83}\ , \left|\overrightarrow{BC}\right| = \sqrt{14}$$

$$\boxed{\Longrightarrow The\ distance(D)\ from\ the\ point\ A\ to\ the\ line\ BC\ is\ \frac{\sqrt{83}}{\sqrt{14}}}$$

SP. 604. Let be $A(2,1,0); B(1,2,1); C(3,3,3); D(0,0,4)$. Find the distance from the point D to the real plane (ABC).

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$$\overrightarrow{AB} = \vec{B} - \vec{A} = (-1,1,1)\ , \overrightarrow{AC} = \vec{C} - \vec{A} = (1,2,3)$$

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = (1, 4, -3)$$

$$Point - normal\ equation: a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$1(x - 2) + 4(y - 1) + (-3)(z - 0) = 0 \Rightarrow x + 4y - 3z - 6 = 0$$

$$\Rightarrow Distance\ d\ from\ a\ point\ (x_0, y_0, z_0)\ to\ a\ plane\ ax + by + cz + d_0 = 0$$

$$d = \frac{|ax_0 + by_0 + cz_0 + d_0|}{\sqrt{a^2 + b^2 + c^2}} = \frac{18}{\sqrt{26}}$$

$$:The\ distance\ d\ from\ the\ point\ D\ to\ the\ plane\ (ABC)\ is\ \frac{18}{\sqrt{26}}$$

JP.607 Solve for reals:

$$\sqrt[4]{8x - 7} + \sqrt[4]{3 - 2x^4} = 2$$

Proposed by Marin Chirciu – Romania

Solution by Omkumar Rao – India

$$x \geq \frac{7}{8}, x^4 \leq \frac{3}{2}\ let\ m = \sqrt[4]{8x - 7}, n = \sqrt[4]{3 - 2x^4}, m + n = 2$$

$$m^4 + n^4 \geq \frac{(m + n)^4}{8} = 2, Equality\ iff\ m = n.$$

$$m^4 + n^4 = 8x - 7 + 3 - 2x^4 = -2x^4 + 8x - 4$$

$$m^4 + n^4 \geq 2 \Rightarrow -2x^4 + 8x - 4 \geq 2 \Leftrightarrow x^4 - 4x + 3 \leq 0 \Rightarrow (1)$$

$$x^4 - 4x + 3 = (x - 1)^2(x^2 + 2x + 3) \geq 0\ for\ all\ real\ x. \Rightarrow (2)$$

$$From\ (1)\ and\ (2), x^4 - 4x + 3 = 0 \Rightarrow x = 1$$

SP.596. Solve for real numbers:

$$\begin{cases} (\sin x)^2 = \frac{1}{2} + (\sin(y - z))^2 \\ (\sin y)^2 = \frac{1}{3} + (\sin(z - x))^2 \\ (\sin z)^2 = \frac{1}{6} + (\sin(x - y))^2 \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$$A = y + z - x\,, B = z + x - y\,, \qquad C = x + y - z$$

$$(\sin x)^2 - (\sin(y-z))^2 = \sin(B)\sin(C) \Rightarrow \frac{1}{2}$$

$$(\sin y)^2 - (\sin(z-x))^2 = \sin(C)\sin(A) \Rightarrow \frac{1}{3}$$

$$(\sin z)^2 - (\sin(x-y))^2 = \sin(A)\sin(B) \Rightarrow \frac{1}{6}$$

$$\sin(A)\sin(B)\sin(C) \Rightarrow \pm\frac{1}{6}$$

$$\sin(A) = \pm\frac{1}{3}\,, \sin(B) = \pm\frac{1}{2}\,\,, \sin(C) = \pm 1$$

$$(x, y, z) = \left(\frac{B+C}{2}, \frac{A+C}{2}, \frac{A+B}{2}\right)\,\, , m_1, m_2, m_3 \in Z.$$

$$\boxed{(A, B, C) = \left(\pm\sin^{-1}\frac{1}{3} + 2m_1\pi, \pm\sin^{-1}\frac{1}{2} + 2m_2\pi, \pm\frac{\pi}{2} + 2m_3\pi\right)}$$

SP. 599 We consider the function $f: D \to R$

$$f(x) = x\int_{x}^{x+\frac{3}{x}} t\sin^{-1}\left(\frac{1}{t}\right)dt,$$

where D is the maximal domain of the function.

a. Find the domain D.

b. Show that the function $f(x)$ is even.

c. Calculate $\lim\limits_{x\to-\infty} f(x)$

Proposed by Vasile Mircea Popa – Romania

Solution by Omkumar Rao – India

(a) ***Domain D***:

$$For\ \sin^{-1}\left(\frac{1}{t}\right)\ to\ be\ defined, \left|\frac{1}{t}\right| \le 1 \Rightarrow |t| \ge 1$$

$$\Rightarrow If\ x > 0, x + \frac{3}{x} > x.\ (x \ge 1)\ so\ that\ \left[x, x + \frac{3}{x}\right] \in [\![1,\infty)$$

$$\Rightarrow If\ x < 0, x + \frac{3}{x} < x.\ (x \le -1)\ so\ that\ \left[x + \frac{3}{x}, x\right] \in (-\infty, -1]$$

$$\boxed{D = (-\infty, -1] \cup [\![1,\infty)}$$

(b) *Proof that* $f(x)$ *is even.*

$$f(-x) = (-x)\int_{-x}^{-x-\frac{3}{x}} t\sin^{-1}\left(\frac{1}{t}\right)dt \overset{x\to -x}{\Longrightarrow} x\int_{x}^{x+\frac{3}{x}} t\sin^{-1}\left(\frac{1}{t}\right)dt = f(x)$$

$$\boxed{f(-x) = f(x)\ \ \forall x \in D}$$

$$(c)\ \lim_{x\to-\infty} f(x)$$

$$Since\ f(x)\ is\ even, \lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x)$$

$$t\sin^{-1}\left(\frac{1}{t}\right) = t\left(\frac{1}{t} + O\left(\frac{1}{t^3}\right)\right) = 1 + O\left(\frac{1}{t^2}\right)$$

$$\int_{x}^{x+\frac{3}{x}} t\sin^{-1}\left(\frac{1}{t}\right)dt = \int_{x}^{x+\frac{3}{x}} \left(1 + O\left(\frac{1}{t^2}\right)\right)dt = \frac{3}{x} + O\left(\frac{1}{x^3}\right)$$

$$\boxed{\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(3 + O\left(\frac{1}{x^2}\right)\right) = 3}$$

UP.589. *If* $X, Y, Z \in M_4(C)$ *are matrices such that:*

$$\begin{cases} X = 2Y + Z \\ X^2 = 4Y + 4Z \\ X^3 = 8Y + 12Z \end{cases} \quad then: X^{2024} = 2^{2024}Y + 2024.2^{2023}.Z$$

Proposed by Daniel SItaru – Romania

Solution by Omkumar Rao – India

$$X = 2Y + Z \Rightarrow Z = X - 2Y$$

$$X^2 = 4Y + 4Z \Longrightarrow X^2 = 4Y + 4(X - 2Y) \Longrightarrow 4Y = 4X - X^2$$

$$X^3 = 8Y + 12Z = 8Y + 12(X - 2Y) = 12X - 16Y$$

$$X^3 - 4X^2 + 4X = 0 \Longrightarrow X(X - 2)^2 = 0$$

$$X^n = K_n X + P_n X^2$$

$$X^n = K_n(2Y + Z) + P_n(4Y + 4Z)$$

$$2^n = 2K_n + 4P_n \ , \ n.2^{n-1} = K_n + 4P_n \Longrightarrow K_n = (2 - n)2^n \ , P_n = (n - 1)2^{n-1}$$

$$X^n = (2 - n)2^n(2Y + Z) + (n - 1)2^{n-1}(4Y + 4Z)$$

$$X^n = 2^n Y + n.2^{n-1}Z$$

$$\boxed{For\ n = 2024 \ , X^{2024} = 2^{2024}Y + 2024.2^{2023}.Z}$$

UP.590. *If $A, B \in M_4(R); A.B = B.A$ Then:*

$$det\left(A^4 + B^4 + AB(A^2 + AB + B^2)\right) \geq 0$$

Proposed by Daniel Sitaru – Romania

Solution by Omkumar Rao – India

$$\Longrightarrow det\left(\frac{A^5 - B^5}{A - B}\right)$$

$$= det(A - \omega B).det(A - \varpi B).det(A - \omega^2 B).det(A - \varpi^2 B)$$

$$= det(A - \varpi B) = \overline{det(A - \omega B)}$$

$$= |det(A - \omega B)|^2 \left|det(A - \omega^2 B)\right|^2 \geq 0$$

$$\boxed{\Longrightarrow det\left(A^4 + B^4 + AB(A^2 + AB + B^2)\right) \geq 0}$$

UP.596. *If $x > 0, y > 0, z > 0$ prove that there exists $u > 0$ such as* $\dfrac{\sin x \sin y + \sin y \sin z + \sin z \sin x}{xy + yz + zx} = \dfrac{\sin u}{u}$

Proposed by Cristian Miu – Romania

Solution by Omkumar Rao – India

$$Since\ x, y, z > 0 \ , 0 < \sin x \sin y < xy$$

$$0 < \sin y \sin z < yz, 0 < \sin z \sin x < zx$$

$$0 < \sin x \sin y + \sin y \sin z + \sin z \sin x < xy + yz + zx$$

$$0 < \frac{\sin x \sin y + \sin y \sin z + \sin z \sin x}{xy + yz + zx} < 1$$

$$Let\ f(u) = \frac{\sin u}{u}, f(u) = \frac{\sin u}{u}\ is\ continuo\ us\ on\ (0, \pi):$$

$$\lim_{u \to 0^+} f(u) = 1\ , \qquad f(\pi) = 0$$

$$By\ Intermediate\ Value\ Theorem\ , there\ exists\ at\ least\ one\ u \in (0, \pi)\ such\ that: \frac{\sin x \sin y + \sin y \sin z + \sin z \sin x}{xy + yz + zx} = \frac{\sin u}{u}$$

SP.591. *If* $a, b, c > 0$, $a^8 + b^8 + c^8 \leq 768$ *then*:

$$\sum \frac{1}{\sqrt{4 + a^5}} \geq \frac{1}{2}$$

Proposed by Marin Chirciu – Romania

Solution by Omkumar Rao – India

$$\overset{Bergstorm's\ Inequality}{\Rightarrow} \sum \frac{1}{\sqrt{4 + a^5}} \geq \frac{9}{\sqrt{4 + a^5} + \sqrt{4 + b^5} + \sqrt{4 + c^5}}$$

$$\left(\sqrt{4 + a^5} + \sqrt{4 + b^5} + \sqrt{4 + c^5}\right)^2 \leq (9)\left(12 + a^5 + b^5 + c^5\right)$$

$$\overset{Power-Mean\ Inequality}{\Rightarrow} \left(\frac{a^5 + b^5 + c^5}{3}\right)^{\frac{1}{5}} \leq \left(\frac{a^8 + b^8 + c^8}{3}\right)^{\frac{1}{8}}$$

$$\left(\frac{a^5 + b^5 + c^5}{3}\right)^{\frac{1}{5}} \leq (256)^{\frac{1}{8}} = 2$$

$$a^5 + b^5 + c^5 \leq 96$$

$$\left(\sqrt{4 + a^5} + \sqrt{4 + b^5} + \sqrt{4 + c^5}\right)^2 \leq (9)(12 + 96) = 324$$

$$\sum \frac{1}{\sqrt{4 + a^5}} \geq \frac{9}{18} = \frac{1}{2}$$