

RMM - Triangle Marathon 4301 - 4400

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4301. In $\triangle ABC$, let D be a point in its interior such that: $\angle BAD = \angle DBC = \angle DCA$, $\angle ADC = 90^\circ$ and $AB \cdot AC = \frac{\sqrt{3}}{4} BC^2$. Find $\cos(\angle BAD)$.

Proposed by Adgozel Adgozalov-Azerbaijan

Solution by Sahel Koley-Tarakeswar-India

Given that $\angle BAD = \angle DBC = \angle DCA$. Suppose: $\angle BAD = \angle DBC = \angle DCA = \alpha$, also $\angle DAC = 90^\circ$ and $AB \cdot AC = \frac{\sqrt{3}}{4} BC^2$ [Assume that, $BC = a, AC = b$ and $AB = c$] as, $\angle DAC = 90^\circ$, So from $\triangle ADC$ we get $\angle DCA = 90^\circ - \alpha$. and also from $\triangle BAC$, we get, $\angle BAC = \alpha + 90^\circ - \alpha = 90^\circ$

So, $\triangle BAC$ is a right angled triangle

$$AB \cdot AC = \frac{\sqrt{3}}{4} BC^2 \Rightarrow bc = \frac{\sqrt{3}}{4} a^2 \Rightarrow (a\sin B)(a\cos B) = \frac{\sqrt{3}}{4} a^2$$

[since, $\angle BAC = 90^\circ, \frac{b}{a} = \sin B, \frac{c}{a} = \cos B$] $\Rightarrow \frac{a^2}{2}[2\sin B \cos B] = \frac{\sqrt{3}}{4} a^2 \Rightarrow \sin 2B = \frac{\sqrt{3}}{2}$

So, $\angle B = 30^\circ$ or 60°, Hence, $\angle ABD = \angle B - \alpha, AD = b\sin\alpha$

From $\triangle ABD$, we get: $\angle ADB = 180^\circ - \alpha - \angle B - \alpha = 180^\circ - \angle B$

$$\frac{AB}{\sin\angle ADB} = \frac{AD}{\sin\angle ABD} \Rightarrow \frac{c}{\sin B} = \frac{b\sin\alpha}{\sin(B-\alpha)} \Rightarrow \frac{b}{c}\sin B \sin\alpha = \sin(B-\alpha)$$

$$\Rightarrow \tan B \sin B \sin\alpha = \sin B \cos\alpha - \cos B \sin\alpha$$

$$\Rightarrow \tan B \sin\alpha = \cos\alpha - \cot B \sin\alpha \Rightarrow \sin\alpha(\tan B + \cot B) = \cos\alpha$$

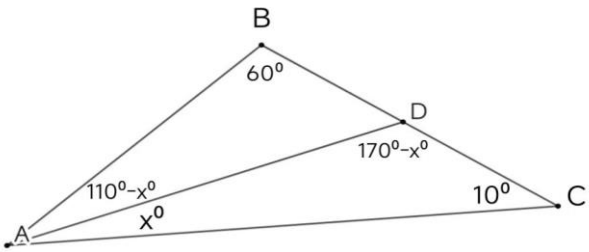
$$\Rightarrow \tan\alpha = \frac{1}{\tan B + \cot B} = \sin B \cos B = \frac{\sqrt{3}}{4}$$

$$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{1+\frac{3}{16}}} = \frac{1}{\sqrt{\frac{19}{16}}} = \frac{4\sqrt{19}}{19}$$

4302. In ΔABC, let $\angle B = 60^\circ$ and $\angle C = 10^\circ$. Point $D \in BC$ such that $|AC| = \sqrt{3} \cdot |BD|$, and $\angle DAC = x^\circ$, then find x.

Proposed by Adgozel Adgozalov-Azerbaijan

Solution by Mirsadix Muzefferov-Azerbaijan



in ΔABD sine rule $\frac{AD}{\sin 60^\circ}=\frac{BD}{\sin(110^\circ-x)}$

in ΔADC sine rule $\frac{AD}{\sin 10^\circ}=\frac{AC}{\sin(170^\circ-x)}$

$$\Rightarrow \begin{cases} \frac{2AD}{\sqrt{3}}=\frac{BD}{\cos(20^\circ-x)} \\ \frac{\sin 10^\circ}{AD}=\frac{\sin(10^\circ+x)}{AC} \end{cases} \Rightarrow 2\cdot\sin 10^\circ\cdot\cos(20^\circ=\sin(10^\circ+x)$$

$$\Rightarrow 2\cdot\frac{1}{2}(\sin(30^\circ-x)+\sin(x-10^\circ))=\sin(x+10^\circ)$$

$$\Rightarrow \sin(30^\circ-x)=\sin(x+10^\circ)-\sin(x-10^\circ)\Rightarrow \sin(30^\circ-x)=2\sin 10^\circ\cdot\cos x$$

$$\Rightarrow \sin 30^\circ\cdot\cos x-\cos 30^\circ\cdot\sin x=2\sin 10^\circ\cdot\cos x\Rightarrow \frac{\cos x}{2}-\frac{\sqrt{3}}{2}\sin x$$
$$=2\sin 10^\circ\cdot\cos x$$

$$\Rightarrow \tan x=\frac{1-4\sin 10^\circ}{\sqrt{3}}$$ · let's prove that : $1-4\sin 10^\circ=\sqrt{3}\tan 10^\circ$

$$(1-4\sin 10^\circ)\cdot\cos 10^\circ=\sqrt{3}\sin 10^\circ\Rightarrow \cos 10^\circ-2\sin 20^\circ=\sqrt{3}\sin 10^\circ$$

$$\Rightarrow \cos 10^\circ-\sqrt{3}\sin 10^\circ=2\sin 20^\circ\Rightarrow \frac{1}{2}\cdot\cos 10^\circ-\frac{\sqrt{3}}{2}\sin 10^\circ=\sin 20^\circ$$

$$\Rightarrow \sin 30^\circ\cdot\cos 10^\circ-\cos 30^\circ\cdot\sin 10^\circ=\sin 20^\circ\Rightarrow \sin(30^\circ-10^\circ)=\sin 20^\circ$$

$$\Rightarrow \sin 20^\circ=\sin 20^\circ \quad \text{(true). So } x=10^\circ$$

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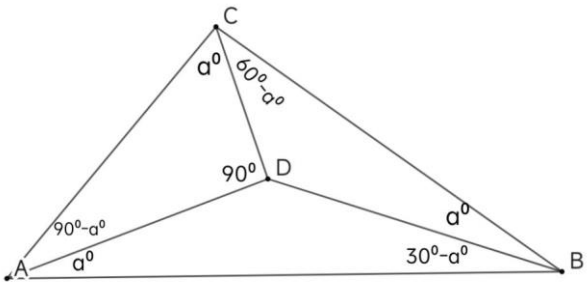
4303. In $\triangle ABC$, let D be a point in its interior such that:

$$\angle BAD = \angle DBC = \angle DCA = \alpha, \quad \angle ADC = 90^\circ \quad \text{and} \quad AB \cdot AC = \frac{\sqrt{3}}{4} BC^2.$$

Find $\cos(\angle BAD)$.

Proposed by Adgozel Adgozalov–Azerbaijan

Solution by Mirsadix Muzefferov–Azerbaijan



In $\triangle ABC$ ($\angle A = 90^\circ$), by Pythagorean theorem: $AC^2 + AB^2 = BC^2$

$$AC^2 + AB^2 = \frac{4 \cdot AB \cdot AC}{\sqrt{3}} \Rightarrow \sqrt{3} \cdot \left(\frac{AB}{AC}\right)^2 - 4\left(\frac{AB}{AC}\right) + \sqrt{3} = 0, \qquad \frac{AB}{AC} = \frac{2 \pm 1}{\sqrt{3}}$$

$$1)\ \frac{AB}{AC} = \sqrt{3} \qquad 2)\ \frac{AB}{AC} = \frac{1}{\sqrt{3}}$$

1) Case: $\frac{AB}{AC} = \sqrt{3} \Rightarrow \angle C = 60^\circ,\ \angle B = 30^\circ$

Let's use Ceva's theorem in $\triangle ABC$:

$$\sin\alpha \cdot \sin\alpha \cdot \sin\alpha = \sin(30^\circ - \alpha) \cdot \sin(60^\circ - \alpha) \cdot \sin(90^\circ - \alpha)$$

$$\sin^3\alpha = \cos\alpha \cdot \sin(30^\circ - \alpha) \cdot \sin(60^\circ - \alpha)$$

$$\frac{\sin^3\alpha}{\cos\alpha} = \frac{1}{2}(\cos 30^\circ - \cos(90^\circ - 2\alpha))$$

$$\frac{2\sin^3\alpha}{\cos\alpha} = \frac{\sqrt{3}}{2} - \sin 2\alpha \Rightarrow \frac{2\sin^3\alpha}{\cos\alpha} + 2\sin\alpha\cos\alpha = \frac{\sqrt{3}}{2}$$

$$2\sin^3\alpha + 2\sin\alpha \cdot \cos^2\alpha = \frac{\sqrt{3}}{2}\cos\alpha \Rightarrow 2\sin\alpha(\sin^2\alpha + \cos^2\alpha) = \frac{\sqrt{3}}{2}\cos\alpha$$

$$\tan\alpha=\frac{\sqrt{3}}{4}\Rightarrow\cos\alpha=\frac{4\sqrt{19}}{19}$$

The 2) $\frac{AB}{AC}=\frac{1}{\sqrt{3}}$ second case is found in the same way.

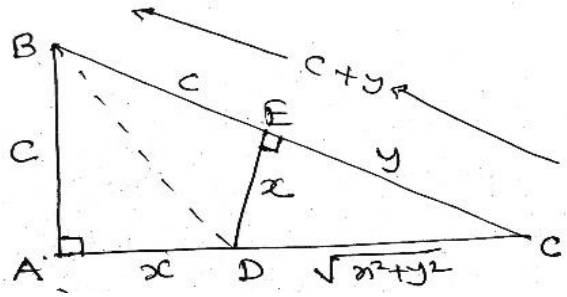
4304. ΔABC *with* $\angle A=90°, D\in(AC), E\in(BC), AD=DE,$ $AB=BE, DE$ *perpendicular on* BC & *perimeter* $(ABED)=$ *perimeter* (DCE). *Prove that*:

$$\frac{AB}{DE}=\text{golden ratio}$$

Proposed by Neculai Stanciu-Romania

Solution by Tapas Das-India

Let $AD=DE=x$ & $AB=BE=c, BC=c+y,$ *In* ΔCDE , $\angle DEC=90°$ *then*



$$CD=\sqrt{x^2+y^2}$$

$$\text{perimeter}\,(ABED)=\text{perimeter}\,(DCE)$$

$$or, c+c+x+x=x+y+\sqrt{x^2+y^2}\ or, 2c+x-y=\sqrt{x^2+y^2}\ (1)$$

$$\Delta ABC\ we\ get\ (c+y)^2=c^2+\left(x+\sqrt{x^2+y^2}\right)^2\ or, c^2+2cy+y^2$$

$$=c^2+x^2+x^2+y^2+2x\sqrt{x^2+y^2}$$

$$cy=x^2+x\sqrt{x^2+y^2}\overset{(1)}{=}x^2+x(2c+x-y)\ or\ y(c+x)=2x(c+x)$$

$$now\ from(1)we\ have:$$

$$2c+x-y=\sqrt{x^2+y^2}\ or\ 2c+x-2x\overset{y=2x}{=}\sqrt{x^2+(2x)^2}\ or, 2c-x=x\sqrt{5}\ or$$

$$\frac{c}{x}=\frac{\sqrt{5}+1}{2},\qquad so\ \frac{AB}{DE}=\frac{c}{x}=\frac{\sqrt{5}+1}{2}=\text{golden ratio}=\phi$$

4305. If in $\triangle ABC$ with $D\in(AC), E\in(BC)$ such that $AB=CD, DB=DE$ and $\angle ABD=\angle EDC=2\cdot\angle ACB$, then find the angle measures of triangle ABC

Proposed by Neculai Stanciu-Romania

Solution by Nawar Alasadi-Iraq

$$\begin{aligned}
\angle C &= \alpha \Rightarrow \angle ABD = \angle EDC = 2\alpha \\
\angle DEB &= \angle C + \angle EDC = \alpha + 2\alpha = 3\alpha \\
DB &= DE \Rightarrow \angle DBE = \angle DEB = 3\alpha \\
\angle B &= \angle ABD + \angle DBE = 2\alpha + 3\alpha = 5\alpha \\
\angle BDA &= \angle C + \angle DBC = \alpha + 3\alpha = 4\alpha \\
\angle A &= 180^\circ - (\angle B + \angle C) = 180^\circ - 6\alpha
\end{aligned}$$

Law of Sines in $\triangle ABD$: $\dfrac{AB}{\sin(4\alpha)} = \dfrac{BD}{\sin(180^\circ - 6\alpha)} \Rightarrow AB = BD\dfrac{\sin(4\alpha)}{\sin(6\alpha)}$

Law of Sines in $\triangle BCD$: $\dfrac{CD}{\sin(3\alpha)} = \dfrac{BD}{\sin(\alpha)} \Rightarrow CD = BD\dfrac{\sin(3\alpha)}{\sin(\alpha)}$

$$\begin{aligned}
AB &= CD \\
BD\frac{\sin(4\alpha)}{\sin(6\alpha)} &= BD\frac{\sin(3\alpha)}{\sin(\alpha)} \\
\sin(4\alpha)\sin(\alpha) &= \sin(6\alpha)\sin(3\alpha) \\
\frac{1}{2}[\cos(3\alpha) - \cos(5\alpha)] &= \frac{1}{2}[\cos(3\alpha) - \cos(9\alpha)] \\
\cos(5\alpha) &= \cos(9\alpha)
\end{aligned}$$

$$\angle A > 0 \Rightarrow 180^\circ - 6\alpha > 0 \Rightarrow \alpha < 30^\circ$$

$$9\alpha + 5\alpha = 360^\circ \Rightarrow 14\alpha = 360^\circ \Rightarrow \alpha = \frac{180^\circ}{7}$$

$$\begin{aligned}
\angle A &= 180^\circ - 6\left(\frac{180^\circ}{7}\right) = \frac{180^\circ}{7} = \frac{\pi}{7} \\
\angle B &= 5\left(\frac{180^\circ}{7}\right) = \frac{900^\circ}{7} = \frac{5\pi}{7} \\
\angle C &= \frac{180^\circ}{7} = \frac{\pi}{7}
\end{aligned}$$

4306. In any Δ ABC the following relationship holds :

$$\sqrt{\frac{R}{2r}}\cdot\left(3s - \sum_{cyc}\left(p_a\cdot\sqrt{\frac{l_a}{g_a}}\right)\right) \geq 2\sum_{cyc}\frac{g_b g_c}{n_a + s}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$m_a n_a \overset{?}{\geq} p_a^2 + \frac{(b-c)^2}{18} \overset{\text{Fustei and Ajiba}}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s(b-c)^2}{a}\right)\overset{?}{\geq}\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$
$$+\frac{(b-c)^4}{324}+\frac{(b-c)^2}{9}.\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$
$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a}+\frac{1}{4}\right)+\frac{s(b-c)^4}{4a}\overset{?}{\geq}\frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4}+$$
$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$
$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$
$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2\overset{?}{\geq}0\ (\because (b-c)^2\geq 0)$$
$$\Leftrightarrow\frac{s(s-a)(144s^3-52s^2a-16sa^2+5a^3)}{36a(2s+a)^2}+$$
$$\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2\overset{?}{\geq}0$$
$$\Leftrightarrow\frac{s(s-a)\left((s-a)(144s^2+92sa+76a^2)+81a^3\right)}{36a(2s+a)^2}+$$
$$\frac{(s-a)\left((s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$
$$\overset{?}{\geq}0\rightarrow\text{true (strict inequality)}\ \therefore m_an_a\geq p_a^2+\frac{(b-c)^2}{18}\geq p_a^2\Rightarrow m_an_a\geq p_a^2\rightarrow(1)$$

Bogdan Fustei

$$\text{and } n_ag_a\geq m_al_a\rightarrow(2)\ \therefore (1)\bullet(2)\Rightarrow(m_an_a)(n_ag_a)\geq p_a^2.m_al_a$$
$$\Rightarrow n_a\geq p_a.\sqrt{\frac{l_a}{g_a}}\Rightarrow\sqrt{\frac{R}{2r}}.\left(s-p_a.\sqrt{\frac{l_a}{g_a}}\right)(s+n_a)\geq\sqrt{\frac{R}{2r}}.(s^2-n_a^2)\overset{\text{Bogdan Fustei}}{=}$$
$$\sqrt{\frac{R}{2r}}.\frac{2h_ar_aw_a}{w_a}\geq\sqrt{\frac{R}{2r}}.\frac{2h_ag_bg_c}{w_a}$$

(Reference : Solution to Inequality in Triangle − 127
by Bogdan Fustei; published at www.ssmrmh.ro)

$$\therefore\sqrt{\frac{R}{2r}}.\left(s-p_a.\sqrt{\frac{l_a}{g_a}}\right)(s+n_a)\overset{(*)}{\geq}\sqrt{\frac{R}{2r}}.\frac{2h_ag_bg_c}{w_a}$$
$$\text{Now, }(b+c)^2\overset{?}{\geq}32Rr.\cos^2\frac{A}{2}=8r(r_b+r_c)=8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$
$$=8(s-a)(s-b)(s-c).\frac{a}{(s-b)(s-c)}=4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2+4a^2-4a(b+c)\overset{?}{\geq}0\Leftrightarrow (b+c-2a)^2\overset{?}{\geq}0\rightarrow true$$

$$\therefore b+c\geq\sqrt{32Rr}.\cos\frac{A}{2}\ and\ analogs\Rightarrow\frac{w_a}{h_a}=\frac{2R.2bc.\cos\frac{A}{2}}{(b+c)bc}\leq\frac{4R.\cos\frac{A}{2}}{\sqrt{32Rr}.\cos\frac{A}{2}}=\sqrt{\frac{R}{2r}}$$

$$\Rightarrow\sqrt{\frac{R}{2r}}.\frac{2h_ag_bg_c}{w_a}\geq 2.\sqrt{\frac{R}{2r}}.\frac{g_bg_c}{\sqrt{\frac{R}{2r}}}\Rightarrow\sqrt{\frac{R}{2r}}.\frac{2h_ag_bg_c}{w_a}\overset{(**)}{\geq}2g_bg_c\therefore(*)\ and\ (**)\Rightarrow$$

$$\sqrt{\frac{R}{2r}}.\left(s-p_a.\sqrt{\frac{l_a}{g_a}}\right)\geq\frac{2g_bg_c}{s+n_a}\ and\ analogs\ and\ so,$$

$$\sqrt{\frac{R}{2r}}.\left(3s-\sum_{cyc}\left(p_a.\sqrt{\frac{l_a}{g_a}}\right)\right)\geq 2\sum_{cyc}\frac{g_bg_c}{n_a+s}\ \forall\Delta ABC,$$

$''='' iff\ \Delta ABC\ is\ equilateral\ (QED)$

4307. In any ΔABC the following relationship holds :

$$\frac{1}{4}\sum_{cyc}\frac{(n_a+s)l_a}{r}\geq\sum_{cyc}\frac{g_bg_c}{a-\sqrt{4r^2+(n_a-g_a)^2}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

Let $x=s-a,y=s-b,z=s-c$ and then : $a=y+z,b=z+x,$
$c=x+y$ and $s=x+y+z$ and furthermore, we denote : $\frac{y+z}{x}=m$ and $\frac{yz}{x^2}=n$
and then, we have the following set $''S''$ of relations : $y^2+z^2=x^2(m^2-2n),$

$$\overset{\text{squaring}}{\underset{\text{and via Bogdan Fustei}}{\Leftrightarrow}}$$

$$y^3+z^3=x^3(m^3-3nn),\ and\ now,\ g_bg_c\overset{?}{\leq}r_al_a$$

$$\left((s-b)^2+\frac{4(s-a)(s-b)(s-c)}{b}\right)\left((s-c)^2+\frac{4(s-a)(s-b)(s-c)}{c}\right)\overset{?}{\leq}$$

$$\frac{s(s-b)(s-c)}{s-a}.\frac{4bc}{(b+c)^2}.s(s-a)$$

$$\Leftrightarrow 4(z+x)^2(x+y)^2(x+y+z)^2\overset{?}{\geq}(4zx+xy+yz)(4xy+zx+yz)(2x+y+z)^2$$

$$\Leftrightarrow 4x^6+16x^5(y+z)+8x^4(y^2+z^2)-12x^4.yz-32x^3.yz(y+z)-$$

$$5x^2.yz(y+z)^2+3x.yz(y^3+z^3)+13x.y^2z^2(y+z)+3y^2z^2(y+z)^2\overset{?}{\geq}0\ (1)\ and\ via$$

set of relations $''S''$, to prove (1), it suffices to prove, following simplification :

$$\overbrace{(3m^2+4m)}^{\Omega_1}n^2+\overbrace{(3m^3-5m^2-32m-28)}^{\Omega_2}n+\overbrace{(8m^2+16m+4)}^{\Omega_3}\overset{?}{\geq}0\ \ (2)$$

and it's trivially true when : $\Omega_2\geq 0$ and when : $\Omega_2<0$, then :

$$3m^3-5m^2-32m-28=(m-5)(3m^2+10m+18)+62<0$$

$$\Rightarrow (m-5)(3m^2+10m+18)<-62<0\Rightarrow m<5\rightarrow \text{(i)}$$

Now, LHS of (2) is a quadratic polynomial in "n" with discriminant, $\boxed{\delta}=$

$$\sigma_2^2-4\sigma_1\sigma_3=(m+1)(m+2)^2(9m^3-75m^2+40m+196)$$ and when :

$9m^3-75m^2+40m+196\leq 0$, then $\delta\leq 0\Rightarrow$ (2) is trivially true and when :

$9m^3-75m^2+40m+196>0$, then : $9m^3-75m^2+40m+196$

$$=\left(3m(m-5)-3m-\frac{86}{3}\right)(3m-7)-\frac{14}{3}>0$$

$$\Rightarrow\left(3m(m-5)-3m-\frac{86}{3}\right)(3m-7)>\frac{14}{3}>0\ \text{and}\ \because m\overset{\text{via (i)}}{\leq}5$$

$$\Rightarrow 3m(m-5)-3m-\frac{86}{3}<0\ \therefore m<\frac{7}{3}\rightarrow \text{(ii)}$$ and so, when : $\delta>0$,

then it suffices to prove : $2\Omega_1.n\overset{?}{\leq}-\Omega_2-\sqrt{\delta}\Leftrightarrow$ and $\because n\overset{\text{AM-GM}}{\leq}\frac{m^2}{4}$

$$\therefore \text{suffices to prove : } \Omega_1.\frac{m^2}{2}\overset{?}{\leq}-\Omega_2-\sqrt{\delta}\Leftrightarrow -(m+2)^2(3m^2-2m-14)\overset{?}{\geq}2.\sqrt{\delta}$$

and $\because 3m^2-2m-14=\left(m+\frac{5}{3}\right)(3m-7)-\frac{7}{3}\overset{\text{via (ii)}}{<}0\ \therefore$ it suffices to prove :

$$(m+2)^4(3m^2-2m-14)^2\overset{?}{\geq}4(m+1)(m+2)^2(9m^3-75m^2+40m+196)$$

$$\Leftrightarrow \boxed{m(m+2)^2(m-2)^2(3m+4)(3m^2+16m+4)\overset{?}{\geq}0}\ \therefore g_bg_c\leq r_al_a\rightarrow(*)$$

and now, $(n_a-g_a)^2\overset{\text{Bogdan Fustei}}{=}(b-c)^2+2s(s-a)-2n_ag_a\overset{\text{Bogdan Fustei}}{\leq}$

$$(b-c)^2+2s(s-a)-2s(s-a)=(b-c)^2$$

$$\Rightarrow\frac{g_bg_c}{a-\sqrt{4r^2+(n_a-g_a)^2}}\leq\frac{g_bg_c}{a-\sqrt{4r^2+(b-c)^2}}\overset{\text{Bogdan Fustei}}{=}\frac{g_bg_c}{a-\frac{an_a}{s}}=\frac{sg_bg_c}{a(s-n_a)}$$

$$\overset{\text{via }(*)}{\leq}\frac{sr_al_a}{a(s-n_a)}=\frac{sr_al_a(s+n_a)}{a(s^2-n_a^2)}\overset{\text{Bogdan Fustei}}{=}\frac{sr_al_a(s+n_a)}{a.2r_ah_a}=\frac{sr_al_a(s+n_a)}{4rs.r_a}$$

$$\therefore\frac{g_bg_c}{a-\sqrt{4r^2+(n_a-g_a)^2}}\leq\frac{l_a(s+n_a)}{4r}\ \text{and analogs}$$

$$\therefore\frac{1}{4}\sum_{cyc}\frac{(n_a+s)l_a}{r}\geq\sum_{cyc}\frac{g_bg_c}{a-\sqrt{4r^2+(n_a-g_a)^2}}\ \forall\,\Delta ABC,$$

"$=$" iff ΔABC is equilateral (QED)

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4308. In any ΔABC the following relationship holds :

$$m_a\cdot\sqrt{\frac{r_a}{h_a}}+m_b\cdot\sqrt{\frac{r_b}{h_b}}+m_c\cdot\sqrt{\frac{r_c}{h_c}}\geq\frac{(n_a+n_b+n_c)(h_a+h_b+h_c)}{r_a+r_b+r_c}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\sum_{cyc}n_a}{4R+r}\overset{CBS}{\leq}\frac{\sqrt{3\sum_{cyc}n_a^2}}{4R+r}\overset{\text{Bogdan Fustei}}{=}\frac{\sqrt{3\sum_{cyc}\left(s^2-\frac{r}{R}\cdot s^2\cdot\sec^2\frac{A}{2}\right)}}{4R+r}$$

$$=\frac{\sqrt{3\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)}}{4R+r}\overset{?}{\leq}\sqrt{\frac{R}{2r}}\overset{\text{squaring}}{\Leftrightarrow}R^2(4R+r)^2$$

$$+6r^2(4R+r)^2\overset{?}{\geq}6r(3R-r)s^2\quad(*)$$

$$\text{Now, } 6r(3R-r)s^2\overset{\text{Rouche}}{\leq}6r(3R-r)\left(2R^2+10Rr-r^2+2(R-2r)\cdot\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\leq}(R^2+6r^2)(4R+r)^2$$

$$\Leftrightarrow R(R-2r)(16R^2+4Rr-63r^2)\overset{?}{\geq}6r(3R-r)\cdot 2(R-2r)\cdot\sqrt{R^2-2Rr}$$

$$\text{and }\because R-2r\overset{\text{Euler}}{\geq}0\therefore\text{it suffices to prove following squaring :}$$

$$R^2(16R^2+4Rr-63r^2)^2\overset{?}{\geq}144r^2(3R-r)^2(R^2-2Rr)$$

$$\Leftrightarrow 256t^5+128t^4-3296t^3+2952t^2+2097t+288\overset{?}{\geq}0\quad\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow\frac{(4t+1)^2}{100}\left(\left(16t+\frac{336}{5}\right)(10t-21)^2+\frac{2340(t-2)+504}{5}\right)\overset{?}{\geq}0\to\text{true}$$

$$\text{(strict inequality) }\because t\overset{\text{Euler}}{\geq}2\Rightarrow(*)\text{ is true}\Rightarrow\frac{\sum_{cyc}n_a}{\sum_{cyc}r_a}\leq\sqrt{\frac{R}{2r}}\to(1)\text{ and now,}$$

$$m_a\cdot\sqrt{\frac{r_a}{h_a}}\overset{\text{Lascu}}{\geq}\frac{b+c}{2}\cdot\cos\frac{A}{2}\cdot\sqrt{\frac{s\tan\frac{A}{2}\cdot 4R\cos\frac{A}{2}\sin\frac{A}{2}}{2rs}}=\sqrt{\frac{R}{2r}}\cdot\frac{(b+c)a}{4R}\text{ and analogs}$$

$$\therefore\sum_{cyc}\left(m_a\cdot\sqrt{\frac{r_a}{h_a}}\right)\geq\sqrt{\frac{R}{2r}}\cdot\sum_{cyc}\frac{(b+c)a}{4R}\overset{\text{via }(1)}{\geq}\frac{\sum_{cyc}n_a}{\sum_{cyc}r_a}\cdot\sum_{cyc}h_a\text{ and so,}$$

$$m_a\cdot\sqrt{\frac{r_a}{h_a}}+m_b\cdot\sqrt{\frac{r_b}{h_b}}+m_c\cdot\sqrt{\frac{r_c}{h_c}}\geq\frac{(n_a+n_b+n_c)(h_a+h_b+h_c)}{r_a+r_b+r_c}\ \forall\ ABC,$$

$$''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

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4309. In any ΔABC the following relationship holds

$$2\left(m_b^2+m_c^2\right)\le 2r_a^2+r_b^2+r_c^2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Case 1 $\hat{A}$ is non $-$ acute and we assign : $S=\sin\frac{A}{2}, C=\cos\frac{B-C}{2}$ and

$$2\left(m_b^2+m_c^2\right)\overset{?}{\le}2r_a^2+r_b^2+r_c^2\Leftrightarrow 2\sum_{cyc}m_a^2-2m_a^2\overset{?}{\le}2\sum_{cyc}r_a^2-\left(r_b^2+r_c^2\right)$$

$$\Leftrightarrow 2\left(\sum_{cyc}r_a^2-\sum_{cyc}m_a^2\right)+2m_a^2\overset{?}{\ge}r_b^2+r_c^2 \quad (*)$$

$$\text{Now, }\sum_{cyc}r_a^2-\sum_{cyc}m_a^2\overset{\text{Gerretsen and Leibnitz}}{\ge}(4R+r)^2-2(4R^2+4Rr+3r^2)-\frac{3}{4}.9R^2$$

$$=\frac{5(R^2-4r^2)}{4}\overset{\text{Euler}}{\ge}0\ \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$2s(s-a)+\frac{(b-c)^2}{2}\overset{?}{\ge}(r_b+r_c)^2-2s(s-a)$$

$$\Leftrightarrow 4.16R^2(1-S^2)\left(\frac{C+S}{2}\right)\left(\frac{C-S}{2}\right)+8R^2S^2(1-C^2)\overset{?}{\ge}16R^2(1-S^2)^2$$

$$\Leftrightarrow (1-C^2)(3S^2-2)\overset{?}{\ge}0\to\text{true}\because 1\ge C=\cos\frac{B-C}{2}\text{ and }\because\frac{A}{2}\ge\frac{\pi}{4}$$

$$\Rightarrow S\ge\frac{1}{\sqrt{2}}>\frac{\sqrt{2}}{\sqrt{3}}\Rightarrow 3S^2-2>0\ \therefore 2\left(m_b^2+m_c^2\right)<2r_a^2+r_b^2+r_c^2$$

and the inequality is strict because $a=b=c$ cannot occur when $\hat{A}$ is non $-$ acute

Case 2 $\hat{A}$ is acute and $2\left(m_b^2+m_c^2\right)\overset{?}{\le}2r_a^2+r_b^2+r_c^2$

$$\Leftrightarrow 4a^2+(b+c)^2-2bc\overset{?}{\le}4r_a^2+2(r_b+r_c)^2-(b+c)^2+a^2$$

$$\Leftrightarrow 3.16R^2.S^2(1-S^2)+2.16R^2.C^2(1-S^2)-8R^2(C^2-S^2)\overset{?}{\le}16R^2S^2(C+S)^2+$$

$$32R^2(1-S^2)^2\Leftrightarrow 3(1-2S^2)C^2-4S^3.C+15S^2-12S^4-4\overset{?}{\ge}0 \quad (**)$$

Now, LHS of $(**)$ is a quadratic polynomial in $''C''$ with discriminant $\delta=$

$$16S^6-12(1-2S^2)(15S^2-12S^4-4)=2(-136x^3+252x^2-138x+24)$$

$$\left(x=S^2<\frac{1}{2};\text{since }\frac{A}{2}<\frac{\pi}{4}\Rightarrow S<\frac{1}{\sqrt{2}}\right)$$

$$=2(-17(1-t)^3+63(1-t)^2-69(1-t)+24)$$

$(\because 0<2x<1,$ hence we can assign $2x=1-t\ (0<t<1))$

$=17t^3+3t^2+(3t-1)^2>0$ and $\because 1-2S^2>0\ \therefore$ in order to prove $(*)$,

it suffices to prove : $6(1-2S^2)C \overset{?}{\geq} 4S^3+\sqrt{\delta}$ AND $6(1-2S^2)C \overset{?}{\geq} 4S^3-\sqrt{\delta}$ ①, ②

Since $1-2S^2>0$ and $C\leq 1\ \therefore 6(1-2S^2)C-4S^3\leq 6-12S^2-4S^3=p$ (say)

and if : $p\leq 0$, then $\because 0<\sqrt{\delta}\ \therefore$ ① is trivially true and when : $p>0$, then : ①

$\Leftrightarrow 16S^6-12(1-2S^2)(15S^2-12S^4-4)\overset{?}{\geq}(6-12S^2-4S^3)^2$

$\Leftrightarrow 12(1-2S^2)(S+1)(3S+1)(2S-1)^2\overset{?}{\geq}0\rightarrow$ true $\because 0<S<\frac{1}{\sqrt{2}}$ $\boxed{\Rightarrow \text{① is true}}$

Again, since $1-2S^2>0$ and $C>S\ \therefore 6(1-2S^2)C>6(1-2S^2)S\overset{?}{>}4S^3-\sqrt{\delta}$

$\Leftrightarrow \sqrt{\delta}\overset{?}{>}2S(8S^2-3)$ and it's trivially true when : $S\leq\sqrt{\frac{3}{8}}\ (\because \sqrt{\delta}>0)$

and when : $S>\sqrt{\frac{3}{8}}$, then : $\sqrt{\delta}\overset{?}{>}2S(8S^2-3)$

$\Leftrightarrow 16S^6-12(1-2S^2)(15S^2-12S^4-4)\overset{?}{>}4S^2(8S^2-3)^2$

$\Leftrightarrow \frac{-88x^3+116x^2-52x+8}{4}\overset{?}{>}0$

$\Leftrightarrow -11(1-t)^3+29(1-t)^2-26(1-t)+8\overset{?}{>}0\Leftrightarrow 7t^2+(2t-1)^2\overset{?}{>}0\rightarrow$ true

$\Rightarrow \boxed{\text{② is true}}\ \therefore$ both ① and ② are true $\Rightarrow (**)$ is true whenever $\hat{A}$ is acute

$\therefore 2\left(m_b^2+m_c^2\right)\leq 2r_a^2+r_b^2+r_c^2$; with equality occurring iff $C=1$ and $S=\frac{1}{2}$

$\Rightarrow$ iff Δ ABC is equilateral and so, $2\left(m_b^2+m_c^2\right)\leq 2r_a^2+r_b^2+r_c^2\ \forall\ \Delta$ ABC,

$''='' $ iff Δ ABC is equilateral (QED)

4310. In ΔABC the following relationship holds:

$$4w_a+3\sqrt{3}a\leq r_a+10R+7r$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$Let\ x=\frac{A}{2}, d=\frac{B-C}{2}\ \&\ \frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}-x\ \&\ B,C>0\ then\ |d|\leq\frac{\pi}{2}-x$$

$$and\ \cos d=\cos\frac{B-C}{2}\geq\cos\left(\frac{\pi}{2}-x\right)=sinx, let, s=sinx, c=cosd\ then$$

$$0<s\leq c\leq 1$$

$$a=2RsinA=4Rsinx.cosx$$

$$r = 4Rsin\left(\frac{A}{2}\right)sin\left(\frac{B}{2}\right)sin\left(\frac{C}{2}\right) = 2Rs(c-s)\ \left(as\ sin\left(\frac{B}{2}\right)sin\left(\frac{C}{2}\right) = \frac{cosd - sinx}{2}\right)$$

$$r_a = 4Rsin\left(\frac{A}{2}\right)cos\left(\frac{B}{2}\right)cos\left(\frac{C}{2}\right) = 2Rs(c+s)\left(as\ cos\left(\frac{B}{2}\right)cos\left(\frac{C}{2}\right) = \frac{(c+s)}{2}\right)$$

$$sinB + sinC \overset{A+B+C=\pi}{=} 2\cos\left(\frac{A}{2}\right).\cos\left(\frac{B-C}{2}\right) = 2c.cosx$$

$$bc = 4R^2 sinB.sinC = 2R^2(\cos(B-C) - \cos(B+C))$$

$$= 2R^2\left(2\cos^2\frac{B-C}{2} - 1 + cosA\right)$$

$$= 2R^2\left(2\cos^2\frac{B-C}{2} - 1 + 1 - 2\sin^2\frac{A}{2}\right) = 4R^2(c^2 - s^2)$$

$$Using\ above\ result, w_a = \frac{2bc\cos\left(\frac{A}{2}\right)}{b+c} = 2R\left(c - \frac{s^2}{c}\right)\ \&\ 3\sqrt{3}a = 12\sqrt{3}Rscosx$$

$$r_a + 10R + 7r = 2Rs(c+s) + 10R + 14Rs(c-s) = R(10 + 16sc - 12s^2)$$

$$We\ need\ to\ show: 4w_a + 3\sqrt{3}a \le r_a + 10R + 7r$$

$$8R\left(c - \frac{s^2}{c}\right) + 12\sqrt{3}Rscosx \le R(10 + 16sc - 12s^2)$$

$$10 - 8c + 16sc - 12s^2 + \frac{8s^2}{c} - 12\sqrt{3}scosx \ge 0$$

$$10 - 8c + 16sc - 12s^2 + \frac{8s^2}{c} - 12\sqrt{3}s\sqrt{1 - \sin^2 x} \ge 0$$

$$10 - 8c + 16sc - 12s^2 + \frac{8s^2}{c} - 12\sqrt{3}s\sqrt{1 - s^2} \ge 0$$

$$Let\ F(s,c) = \ 10 - 8c + 16sc - 12s^2 + \frac{8s^2}{c} - 12\sqrt{3}s\sqrt{1-s^2}$$

$$We\ proved\ that\ 0 < s \le c \le 1, so\ now\ we\ calculate$$

$$F(s,c) - F(s,1) = 8(1-c) - 16s(1-c) + \frac{8s^2(1-c)}{c} = 8(1-c)\left(1 - 2s + \frac{s^2}{c}\right)$$

$$= 8(1-c).\frac{(c-s)^2 + c(1-c)}{c} \ge 0\ as\ \ 0 < s \le c \le 1$$

$$so, F(s,c) - F(s,1) \ge 0\ or, F(s,c) \ge F(s,1), now\ we\ will\ show\ F(s,1) \ge 0,$$

$$F(s,1) \ge 0\ or, 2 + 16s - 4s^2 - 12\sqrt{3}s\sqrt{1-s^2} \ge 0$$

$$2 + 16s - 4s^2 \ge 12\sqrt{3}s\sqrt{1-s^2}\ or, (1 + 8s - 2s^2)^2 \ge 108s^2(1-s^2)$$

$$4s^4 - 32s^3 + 60s^2 + 16s + 1 \ge 0\ or, (2s-1)^2(28s^2 + 20s + 1) \ge 0\ true$$

$$so, F(s,c) \ge F(s,1) \ge 0\ o\acute{r},$$

$$10-8c+16sc-12s^2+\frac{8s^2}{c}-12\sqrt{3}s\sqrt{1-s^2}\geq 0$$
$$or,4w_a+3\sqrt{3}a\leq r_a+10R+7r$$

Equality holds for an equilateral triangle.

4311. In ΔABC the following relationship holds:

$$\frac{1}{r_b+r_c}+\frac{1}{r_a}\geq\frac{\sqrt{3}}{a}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\frac{1}{r_b+r_c}+\frac{1}{r_a}=\frac{(s-b)(s-c)}{aF}+\frac{s-a}{F}=\frac{s^2-2s^2+2as+bc-a^2}{aF}=$$
$$=\frac{2as+bc-s^2-a^2}{aF}=\frac{(a+b+c)a+bc-\frac{(a+b+c)^2}{4}-a^2}{aF}=$$
$$=\frac{4(ab+bc+ca)-(a+b+c)^2}{4aF}=\frac{2\sum ab-\sum a^2}{4aF}$$

$$We\ need\ to\ show:$$
$$\frac{1}{r_b+r_c}+\frac{1}{r_a}\geq\frac{\sqrt{3}}{a}\ or\ \frac{2\sum ab-\sum a^2}{4aF}\geq\frac{\sqrt{3}}{a}$$
$$2\sum ab-\sum a^2\geq 4\sqrt{3}F\ true$$

$$Hadwiger-Finsler\ inequality:$$
$$a^2+b^2+c^2\geq 4\sqrt{3}F+(a-b)^2+(b-c)^2+(c-a)^2$$

$$\sum a^2\geq 4\sqrt{3}F+2\sum a^2-2\sum ab\ or\ 2\sum ab-\sum a^2\geq 4\sqrt{3}F$$

Equality holds for an equilateral triangle.

4312. *In any ΔABC with $a\geq b\geq c$, $d\leq e\leq f$ holds:*

$$d(r_b+r_c-2m_a)+e(r_c+r_a-2m_b)+f(r_a+r_b-2m_c)\geq 0$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$r_a + r_c = \frac{F}{s-a} + \frac{F}{s-b} = F\frac{2s-a-c}{(s-a)(s-c)} = b\sqrt{\frac{s(s-b)}{(s-a)(s-c)}} = bcot\left(\frac{B}{2}\right)$$

$$We\ will\ show:$$
$$bcot\left(\frac{B}{2}\right) \geq 2m_b\ or\ b^2\cot^2\frac{B}{2} \geq 2a^2 + 2c^2 - b^2$$

$$b^2\left(1+\cot^2\frac{B}{2}\right) \geq 2a^2 + 2c^2\ or\ \ b^2 \geq 2\sin^2\frac{B}{2}(a^2+c^2)$$

$$b^2 \geq 2(1-cosB)(a^2+c^2)$$

$$or, b^2 \geq \frac{b^2-(a-c)^2}{2ac}(a^2+c^2)\ or, 2acb^2 \geq b^2(a^2+c^2) - (a-c)^2(a^2+c^2)$$
$$b^2(a-c)^2 - (a-c)^2(a^2+c^2) \geq 0$$

$$(a-c)^2(a^2+c^2-b^2) \geq 0\ True\ as(a \geq b \geq c)$$

$$so\ , r_a + r_c \geq 2m_b\ \&\ similarly\ , r_a + r_b \geq 2m_c\ as\ a \geq b \geq c$$
$$let, x = (r_b + r_c - 2m_a), y = (r_c + r_a - 2m_b) > 0, z = (r_a + r_b - 2m_c) > 0\ then$$

$$x + y + z = 2(r_a + r_b + r_c) - 2(m_a + m_b + m_c) \overset{Leuanberger}{\geq}$$
$$\geq\ 2(4R + r) - 2(4R + r) = 0$$

$$d(r_b + r_c - 2m_a) + e(r_c + r_a - 2m_b) + f(r_a + r_b - 2m_c) = dx + ey + fz =$$

$$= d(x+y+z) + (e-d)y + (f-e)z \geq 0\ true\ as\ d \geq 0, e-d \geq 0, f-e \geq 0\ \text{as}$$
$$d \leq e \leq f$$

Equality holds for d=e=f=0.

4313. *In any ΔABC following relationship holds:*

$$\max\left\{\frac{1}{\sqrt{r_a}+\sqrt{r_b}} + \frac{1}{\sqrt{r_a}+\sqrt{r_c}}, \frac{1}{2\sqrt{r_a}} + \frac{1}{\sqrt{r_b}+\sqrt{r_c}}\right\} \leq \frac{2}{\sqrt{h_b}+\sqrt{h_c}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$Let\ x=\frac{1}{\sqrt{r_a}},y=\frac{1}{\sqrt{r_b}},z=\frac{1}{\sqrt{r_c}}$$

$$\frac{1}{r_a}+\frac{1}{r_c}=\frac{1}{F}(s-a+s-c)=\frac{b}{F}=\frac{2}{h_b}\ or,\frac{1}{h_b}=\frac{1}{2}\left(\frac{1}{r_a}+\frac{1}{r_c}\right)=\frac{x^2+z^2}{2}$$

$$\sqrt{h_b}=\frac{\sqrt{2}}{\sqrt{x^2+z^2}}.\ Similarly:\ \frac{1}{r_a}+\frac{1}{r_b}=\frac{2}{h_c}\ or\ \sqrt{h_c}=\frac{\sqrt{2}}{\sqrt{x^2+y^2}}$$

$$Let\ T_1=\frac{1}{\sqrt{r_a}+\sqrt{r_c}}+\frac{1}{\sqrt{r_a}+\sqrt{r_b}}=\frac{xy}{x+y}+\frac{xz}{x+z}=$$

$$=\left(x-\frac{x^2}{x+y}\right)+\left(x-\frac{x^2}{x+z}\right)=2x-x^2\left(\frac{1}{x+y}+\frac{1}{x+z}\right)\&$$

$$T_2=\frac{1}{2\sqrt{r_a}}+\frac{1}{\sqrt{r_c}+\sqrt{r_b}}=\frac{x}{2}+\frac{yz}{y+z}$$

$$\sqrt{x^2+y^2}\geq\frac{x+y}{\sqrt{2}}\ or,\frac{1}{\sqrt{x^2+y^2}}\leq\frac{\sqrt{2}}{x+y}\ \&\ \frac{1}{\sqrt{x^2+z^2}}\leq\frac{\sqrt{2}}{x+z}$$

$$R.H.S=\frac{2}{\sqrt{h_b}+\sqrt{h_c}}=\frac{\sqrt{2}}{\frac{1}{\sqrt{x^2+z^2}}+\frac{1}{\sqrt{x^2+y^2}}}\geq\frac{\sqrt{2}}{\frac{\sqrt{2}}{x+y}+\frac{\sqrt{2}}{x+z}}=\frac{(x+y)(x+z)}{2x+y+z}$$

$$We\ need\ to\ show:T_1\leq\frac{2}{\sqrt{h_b}+\sqrt{h_c}}$$

$$2x-x^2\left(\frac{1}{x+y}+\frac{1}{x+z}\right)\leq\frac{(x+y)(x+z)}{2x+y+z}$$

$$or,2x-x^2u\overset{u=\left(\frac{1}{x+y}+\frac{1}{x+z}\right)}{\leq}\frac{1}{u}\ or,2xu-x^2u^2\leq1\ or,(xu-1)^2\geq0\ True$$

$$We\ need\ to\ show:T_2\leq\frac{2}{\sqrt{h_b}+\sqrt{h_c}}or,\frac{x}{2}+\frac{yz}{y+z}\leq\frac{(x+y)(x+z)}{2x+y+z}$$

$$or,\frac{yz}{y+z}\leq\frac{x(y+z)+2yz}{2(2x+y+z)}or,2yz(2x+y+z)\leq(y+z)(x(y+z)+2yz)$$

$$4yz\leq(y+z)^2\ or\ \ (y-z)^2\geq0\ \ true$$

Equality holds for an equilateral triangle.

4314. *In any ΔABC, with $a\geq b\geq c, d\geq e\geq f\geq 0$ holds:*

$$d(r_a-m_a)+e(r_b-m_b)+f(r_c-m_c)\geq 0$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$As\ a \geq b \geq c\ then\ r_a \geq r_b \geq r_c\ \&\ m_a \leq m_b \leq m_c$$

$$d(r_a - m_a) + e(r_b - m_b) + f(r_c - m_c) =$$

$$= (dr_a + er_b + fr_c) - (dm_a + em_b + fm_c) \geq$$

$$\overset{\substack{Chebyshev\\ d\geq e\geq f}}{\geq} \frac{1}{3}(d+e+f)(r_a + r_b + r_c) - \frac{1}{3}(d+e+f)(m_a + m_b + m_c) \geq$$

$$\overset{Leuenberger}{\geq} \frac{1}{3}(d+e+f)(4R+r) - \frac{1}{3}(d+e+f)(4R+r) = 0$$

Equality holds for d=e=f=0.

4315. In any ΔABC the following relationship holds :

$$2\sum_{cyc}(w_a bc) \leq \prod_{cyc}(r_a + r_b) \leq \frac{8}{3}\sum_{cyc}(r_a m_b m_c)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$2m_b m_c \overset{?}{\geq} \frac{9}{4}a^2 + h_a^2 - m_b^2 - m_c^2 = \frac{9a^2 - 4a^2 - b^2 - c^2}{4} + \frac{4F^2}{a^2}$$

$$= \frac{a^2(5a^2 - b^2 - c^2) + 2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4}{4a^2}$$

$$= \frac{a^2(4a^2 + b^2 + c^2) - (b^2 - c^2)^2}{4a^2}\ and\ it's\ trivially\ true\ if :$$

$$a^2(4a^2 + b^2 + c^2) - (b^2 - c^2)^2 \leq 0\ and\ when :$$

$$a^2(4a^2 + b^2 + c^2) - (b^2 - c^2)^2 > 0, then\ we\ are\ to\ prove :$$

$$4a^4(2c^2 + 2a^2 - b^2)(2a^2 + 2b^2 - c^2) \overset{?}{\geq} (a^2(4a^2 + b^2 + c^2) - (b^2 - c^2)^2)^2$$

$$\Leftrightarrow -a^4(b^2 - c^2)^2 + 2a^2(b^2 + c^2)(b^2 - c^2)^2 - (b^2 - c^2)^4 \overset{?}{\geq} 0$$

$$\Leftrightarrow (b^2 - c^2)^2(-a^4 + 2a^2(b^2 + c^2) - b^4 - c^4 + 2b^2c^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow (b^2 - c^2)^2.16F^2 \overset{?}{\geq} 0 \rightarrow true\ \therefore 2m_b m_c \geq \frac{9}{4}a^2 + h_a^2 - m_b^2 - m_c^2$$

$$= \frac{6a^2 - \sum_{cyc} a^2}{4} + h_a^2\ and\ analogs \Rightarrow \frac{8}{3}\sum_{cyc}(r_a m_b m_c) \geq$$

$$\frac{4}{3}\sum_{cyc}\left(\left(\frac{6a^2-\sum_{cyc}a^2}{4}+h_a^2\right)r_a\right)=$$

$$\frac{3rs}{2}\cdot\frac{4}{3}\sum_{cyc}\frac{(a^2-s^2)+s^2}{s-a}-\frac{2(s^2-4Rr-r^2)(4R+r)}{3}+$$

$$\frac{rs}{3R^2r^2s}\sum_{cyc}\left(b^2c^2(-s^2+sa+bc)\right)=2rs(-5s)+\frac{2s(4Rr+r^2)}{r^2}-$$

$$\frac{2(s^2-4Rr-r^2)(4R+r)}{3}+$$

$$\frac{-s^2((s^2+4Rr+r^2)^2-16Rrs^2)+4Rrs^2(s^2+4Rr+r^2)+(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)}{3R^2r}$$

$$=\frac{24R^2(R-r)s^2-2R^2(4R+r)(s^2-4Rr-r^2)+r(s^4-(4Rr-2r^2)s^2+r(4R+r)^3)}{3R^2}$$

$$\overset{?}{\geq}\prod_{cyc}(r_a+r_b)=64R^3\cdot\frac{s^2}{16R^2}\Leftrightarrow rs^4+(4R^3-26R^2r-4Rr^2+2r^3)s^2+$$

$$r(32R^4+80R^3r+50R^2r^2+12Rr^3+r^4)\overset{?}{\geq}\textcircled{1}\ 0\ \text{and}\ \because(s^2-16Rr+5r^2)^2$$

$$\overset{\text{Gerretsen}}{\geq}0\ \therefore\ \text{in order to prove }\textcircled{1}\text{, it suffices to prove :}$$

$$\text{LHS of }\textcircled{1}\overset{?}{\geq}r(s^2-16Rr+5r^2)^2\Leftrightarrow(2R^3-13R^2r+14Rr^2-4r^3)s^2+$$

$$r(16R^4+40R^3r-103R^2r^2+86Rr^3-12r^4)\overset{?}{\geq}\textcircled{2}\ 0$$

$$\text{Now, }16R^4+40R^3r-103R^2r^2+86Rr^3-12r^4=$$

$$(R-2r)(16R^3+72R^2r+41Rr^2+168r^3)+324r^4\overset{\text{Euler}}{\geq}324r^4>0\ \text{and so,}$$

$$\text{if : }2R^3-13R^2r+14Rr^2-4r^3\geq0\text{, then : }\textcircled{2}\text{ is trivially true and when :}$$

$$2R^3-13R^2r+14Rr^2-4r^3<0\text{, then : LHS of }\textcircled{2}\overset{\text{Gerretsen}}{\geq}$$

$$(2R^3-13R^2r+14Rr^2-4r^3)(4R^2+4Rr+3r^2)+$$

$$r(16R^4+40R^3r-103R^2r^2+86Rr^3-12r^4)\overset{?}{\geq}0$$

$$\Leftrightarrow4t^5-14t^4+25t^3-51t^2+56t-12\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(t^4+3t^3(t-2)+13t(t-2)+t+6)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\Rightarrow\textcircled{2}\Rightarrow\textcircled{1}\ \text{is true and again,}$$

$$2\sum_{cyc}(w_abc)\overset{\text{Lascu + AM-GM}}{\leq}2\sum_{cyc}\left(\sqrt{s(s-a)}.bc\right)=2\sum_{cyc}\left(\sqrt{s(s-a)bc}.\sqrt{bc}\right)$$

$$\overset{CBS}{\leq} 2\sqrt{s^2(s^2+4Rr+r^2)-12Rrs^2}.\sqrt{s^2+4Rr+r^2} \overset{?}{\leq} \prod_{cyc}(r_a+r_b) = 64R^3.\frac{s^2}{16R^2}$$

$$\Leftrightarrow -s^4+(4R^2+4Rr-2r^2)s^2+r^2(32R^2+4Rr-r^2) \overset{?}{\geq} 0 \quad (3)$$

$$\text{Now, LHS of } (3) \overset{Gerretsen}{\geq} -(4R^2+4Rr+3r^2)s^2+(4R^2+4Rr-2r^2)s^2+$$
$$r^2(32R^2+4Rr-r^2) \overset{?}{\geq} 0 \Leftrightarrow 32R^2+4Rr-r^2 \overset{?}{\geq} (4)\ 5s^2 \ \&\ finally, 5s^2 \overset{Gerretsen}{\leq}$$

$$5(4R^2+4Rr+3r^2) \overset{?}{\leq} 32R^2+4Rr-r^2 \Leftrightarrow 4(R-2r)(3R+2r) \overset{?}{\geq} 0 \to true$$

$$\Rightarrow (4) \Rightarrow (3) \text{ is true } \therefore 2\sum_{cyc}(w_a bc) \leq \prod_{cyc}(r_a+r_b) \leq \frac{8}{3}\sum_{cyc}(r_a m_b m_c),$$

$$'' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4316. In ΔABC the following relationship holds:

$$\sum\left(\frac{m_a^{2026}+w_a^{2026}+g_a^{2026}}{3h_a^{2026}}.\frac{r_b r_c}{r^2}\right) \geq 16\frac{R}{r}-5 \geq 13\frac{R}{r}+1 \geq 27$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Tapas Das-India

$$We\ know\ that:$$
$$h_a \leq g_a, h_a \leq m_a, h_a \leq w_a\ then:$$

$$\frac{m_a^{2026}+w_a^{2026}+g_a^{2026}}{3h_a^{2026}} = \frac{1}{3}\left(\left(\frac{m_a}{h_a}\right)^{2026}+\left(\frac{w_a}{h_a}\right)^{2026}+\left(\frac{g_a}{h_a}\right)^{2026}\right) \geq \frac{1}{3}(1+1+1) = 1$$

$$\sum\left(\frac{m_a^{2026}+w_a^{2026}+g_a^{2026}}{3h_a^{2026}}.\frac{r_b r_c}{r^2}\right) \geq \sum\left(.\frac{r_b r_c}{r^2}\right) = \frac{s^2}{r^2} \overset{Gerretsen}{\geq}$$
$$\geq \frac{16Rr-5r^2}{r^2} = 16\frac{R}{r}-5$$

$$16\frac{R}{r}-5 = 13\frac{R}{r}+\frac{3R}{r}-5 \overset{Euler}{\geq} 13\frac{R}{r}+3\times 2-5 = 13\frac{R}{r}+1 \overset{Euler}{\geq} 13\times 2+1 = 27$$

Equality holds for an equilateral triangle.

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4317. In ΔABC the following relationship holds:

$$\frac{r_a}{r_b+r_c}+\frac{r_b}{r_a+r_c}+\frac{r_c}{r_b+r_a}\leq\frac{3R}{4r}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{r_a}{r_b+r_c}+\frac{r_b}{r_a+r_c}+\frac{r_c}{r_b+r_a}\leq\frac{r_a}{4}\left(\frac{1}{r_b}+\frac{1}{r_c}\right)+\frac{r_b}{4}\left(\frac{1}{r_a}+\frac{1}{r_c}\right)+\frac{r_c}{4}\left(\frac{1}{r_b}+\frac{1}{r_a}\right)=$$
$$=\frac{1}{4}\left(\frac{r_a}{r_b}+\frac{r_a}{r_c}+\frac{r_b}{r_c}+\frac{r_b}{r_a}+\frac{r_c}{r_b}+\frac{r_c}{r_a}\right)=\frac{1}{4}\left(\left(\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\right)+\left(\frac{r_b}{r_a}+\frac{r_a}{r_c}+\frac{r_c}{r_b}\right)\right)\overset{Popovici}{\geq}$$
$$\geq\frac{1}{4}\left(\frac{3R}{2r}+\frac{3R}{2r}\right)=\frac{3R}{4r}$$

$$Equality\ holds\ if\ a=b=c$$

4318. In ΔABC the following relationship holds:

$$\sum\left(\frac{m_a^4+w_a^4+g_a^4}{3h_a^4}\cdot\left(\frac{r_a}{r}\right)^2\right)+\frac{ab+bc+ca-4Rr-r^2}{27r^2}\geq 4\left(2\left(\frac{R}{r}\right)^2-1\right)\geq 4\left(\frac{4R}{r}-1\right)\geq 28$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Tapas Das-India

$$m_a\geq h_a, g_a\geq h_a, w_a\geq h_a$$

$$\frac{m_a^4+w_a^4+g_a^4}{3h_a^4}=\frac{1}{3}\left(\left(\frac{m_a}{h_a}\right)^4+\left(\frac{w_a}{h_a}\right)^4+\left(\frac{g_a}{h_a}\right)^4\right)\geq\frac{1}{3}(1+1+1)=1$$

$$\sum\left(\frac{m_a^4+w_a^4+g_a^4}{3h_a^4}\cdot\left(\frac{r_a}{r}\right)^2\right)\geq\sum\left(\left(\frac{r_a}{r}\right)^2\right)=\frac{1}{r^2}((4R+r)^2-2s^2)=$$
$$=\frac{27(4R+r)^2-54s^2}{27r^2}$$

$$\frac{ab+bc+ca-4Rr-r^2}{27r^2}=\frac{s^2+r^2+4Rr-4Rr-r^2}{27r^2}=\frac{s^2}{27r^2}$$

$$\sum\left(\frac{m_a^4+w_a^4+g_a^4}{3h_a^4}\cdot\left(\frac{r_a}{r}\right)^2\right)+\frac{ab+bc+ca-4Rr-r^2}{27r^2}\geq\frac{27(4R+r)^2-54s^2}{27r^2}+\frac{s^2}{27r^2}$$

$$=\frac{27((4R+r)^2-2s^2)}{27r^2}+\frac{s^2}{27r^2}\overset{\substack{Gerretsen\\Mitrinovic}}{\geq}\frac{(4R+r)^2-2(4R^2+4Rr+3r^2)}{r^2}+\frac{27r^2}{27r^2}=$$

$$=\frac{8R^2-5r^2}{r^2}+1=8\left(\frac{R}{r}\right)^2-4=4\left(2\left(\frac{R}{r}\right)^2-1\right)=$$

$$=4\left(\frac{2R}{r}.\frac{R}{r}-1\right)\overset{Euler}{\geq}4\left(2\times2\times\frac{R}{r}-1\right)=4\left(4.\frac{R}{r}-1\right)\overset{Euler}{\geq}4(4\times2-1)=28$$

Equality holds for an equilateral triangle.

4319. In ΔABC the following relationship holds:

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{h_a}}+\frac{\sqrt{b}+\sqrt{c}}{\sqrt{h_b}}+\frac{\sqrt{c}+\sqrt{a}}{\sqrt{h_c}}\geq 2\sqrt[4]{108}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sum_{cyc}\frac{\sqrt{a}+\sqrt{b}}{\sqrt{h_a}}\overset{AM-GM}{\geq}\sum_{cyc}\frac{2\sqrt[4]{ab}}{\sqrt{h_a}}=2\sum_{cyc}\sqrt[4]{\frac{ab}{h_a^2}}\overset{AM-GM}{\geq}$$

$$\geq 2\cdot 3\sqrt[3]{\sqrt[4]{\left(\frac{abc}{h_ah_bh_c}\right)^2}}=6\sqrt[6]{\frac{abc}{h_ah_bh_c}}=6\cdot\sqrt[6]{\frac{4RF}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}}=$$

$$=6\cdot\sqrt[6]{\frac{4RF\cdot abc}{8F^3}}=6\sqrt[6]{\frac{16R^2F^2}{8F^3}}=6\sqrt[6]{\frac{2R^2}{F}}=6\cdot\sqrt[6]{\frac{2R\cdot R}{r\cdot s}}=6\cdot\sqrt[6]{\frac{2R\cdot 2r}{r\cdot s}}=6\cdot\sqrt[6]{\frac{4R}{s}}\geq$$

$$\overset{MITRINOVIC}{\geq}6\cdot\sqrt[6]{\frac{4R}{\frac{3\sqrt{3}}{2}R}}=6\sqrt[6]{\frac{8}{3\sqrt{3}}}=6\sqrt[6]{\left(\frac{2}{\sqrt{3}}\right)^3}=6\sqrt[6]{\frac{2}{\sqrt{3}}}=6\sqrt{\frac{2\sqrt{3}}{3}}=2\cdot\sqrt{9\cdot\frac{2\sqrt{3}}{3}}$$

$$=2\sqrt{6\sqrt{3}}=2\sqrt{\sqrt{36\cdot 3}}=2\sqrt[4]{108}$$

Equality holds for $a=b=c$.

4320. In ΔABC the following relationship holds:

$$\sum\left(\frac{m_a^4+w_a^4+h_a^4}{h_a^4}\cdot\frac{r_a^2}{(s-b)(s-c)}+\frac{(a+b)^2}{4ab}\cdot\frac{r_b+r_c}{r_a}\right)\geq 12.\frac{R}{r}+\sum\frac{r_b+r_c}{r_a}+3\geq 33$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Tapas Das-India

$$\frac{(a+b)^2}{4ab} \overset{AM-GM}{\geq} 1$$

$$\frac{m_a^4+w_a^4+h_a^4}{h_a^4} \geq \frac{h_a^4+h_a^4+h_a^4}{h_a^4} = 3(as\ m_a \geq w_a \geq h_a)$$

$$\sum \frac{r_a^2}{(s-b)(s-c)} = \sum \frac{F^2}{(s-a)^2(s-b)(s-c)} = \sum \frac{s}{s-a} = \frac{s(4R+r)}{sr} = 4\left(\frac{R}{r}\right)+1$$

$$\sum \left(\frac{m_a^4+w_a^4+h_a^4}{h_a^4}\cdot\frac{r_a^2}{(s-b)(s-c)} + \frac{(a+b)^2}{4ab}\cdot\frac{r_b+r_c}{r_a}\right) =$$

$$= \sum \left(\frac{m_a^4+w_a^4+h_a^4}{h_a^4}\cdot\frac{r_a^2}{(s-b)(s-c)}\right) + \sum \left(\frac{(a+b)^2}{4ab}\cdot\frac{r_b+r_c}{r_a}\right) \geq$$

$$\geq \sum \left(3\frac{r_a^2}{(s-b)(s-c)}\right) + \sum \left(1\times\frac{r_b+r_c}{r_a}\right) = 3\left(\frac{4R}{r}+1\right) + \sum \frac{r_b+r_c}{r_a} =$$

$$= 12.\frac{R}{r} + \sum \frac{r_b+r_c}{r_a} + 3 \overset{AM-GM}{\geq} 12.\frac{R}{r} + \sum \frac{2\sqrt{r_b r_c}}{r_a} + 3 \overset{\substack{AM-GM\\ Euler}}{\geq}$$

$$\geq 12\times 2 + 2\times 3 + 3 = 33$$

Equality holds for an equilateral triangle.

4321. In ΔABC the following relationship holds:

$$ar_a + br_b + cr_c \geq 6F$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$ar_a + br_b + cr_c = \sum_{cyc} a\cdot\frac{F}{s-a} = F\sum_{cyc}\frac{a}{s-a} = F\cdot\frac{2(2R-r)}{r} \geq$$

$$\overset{EULER}{\geq} F\cdot\frac{2(4r-r)}{r} = F\cdot\frac{2\cdot 3r}{r} = 6F.$$ Equality holds for $a=b=c$.

Solution 2 by Daniel Sitaru – Romania

$$ar_a + br_b + cr_c \overset{AM-GM}{\geq} 3\sqrt[3]{abcr_a r_b r_c} =$$

$$= 3 \cdot \sqrt[3]{abc \cdot \frac{F}{s-a} \cdot \frac{F}{s-b} \cdot \frac{F}{s-c}} = 3\sqrt[3]{\frac{abcF^3 s}{s(s-a)(s-b)(s-c)}} =$$

$$= 3\sqrt[3]{\frac{4RF \cdot F^3 s}{F^2}} = 3\sqrt[3]{4RsF^2} = 3\sqrt[3]{4Rs \cdot r^2 s^2} = 3 \cdot s\sqrt[3]{4Rr^2} \overset{EULER}{\geq} 3s \cdot \sqrt[3]{4 \cdot 2r \cdot r^2} =$$

$$= 3s \cdot \sqrt[3]{8r^3} = 3s \cdot 2r = 6rs = 6F. \text{ Equality holds for } a = b = c.$$

4322. In any Δ ABC the following relationship holds :

$$\left(\frac{R}{2r}\right)^{\frac{3}{2}} \cdot \sum_{cyc} \frac{r_a^2 + r_b r_c}{w_a(h_b + h_c)} \geq \sum_{cyc} \sqrt{\frac{m_a}{h_a}}$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{r_a^2 + r_b r_c}{w_a(h_b + h_c)} \overset{Lascu + AM-GM}{\geq} \sum_{cyc} \frac{r_a^2 + w_a^2}{\frac{2bc}{b+c}\cos\frac{A}{2} \cdot \frac{a(b+c)}{2R}} \geq \sum_{cyc} \frac{r_a^2 + h_a^2}{4rs \cdot \cos\frac{A}{2}} \overset{Bergstrom}{\geq}$$

$$\frac{1}{4rs} \cdot \left(\frac{(4R+r)^2}{\sum_{cyc}\cos\frac{A}{2}} + \frac{(s^2 + 4Rr + r^2)^2}{4R^2 \cdot \sum_{cyc}\cos\frac{A}{2}} \right) \overset{\substack{\text{Doucet or Trucht} \\ \text{and Gordon} \\ \text{and Jensen}}}{\geq}$$

$$\frac{1}{4rs} \cdot \left(\frac{(4R+r)(\sqrt{3}.s)}{\frac{3\sqrt{3}}{2}} + \frac{(s^2 + 4Rr + r^2)(4\sqrt{3}.sr)}{4R^2 \cdot \frac{3\sqrt{3}}{2}} \right) \overset{Gerretsen}{\geq}$$

$$\frac{1}{6r} \cdot \left(\frac{R^2(4R+r) + r(20Rr - 4r^2)}{R^2} \right)$$

$$\Rightarrow \left(\frac{R}{2r}\right)^{\frac{3}{2}} \cdot \sum_{cyc} \frac{r_a^2 + r_b r_c}{w_a(h_b + h_c)} \geq \left(\frac{R}{2r}\right)^{\frac{3}{2}} \cdot \frac{1}{6r} \cdot \left(\frac{R^2(4R+r) + r(20Rr - 4r^2)}{R^2} \right) \overset{?}{\geq} 3 \cdot \sqrt{\frac{R}{2r}}$$

$$\Leftrightarrow (R^2 - 4r^2)(4R + r) \overset{?}{\geq} 0 \rightarrow \text{true via Euler}$$

$$\therefore \left(\frac{R}{2r}\right)^{\frac{3}{2}} \cdot \sum_{cyc} \frac{r_a^2 + r_b r_c}{w_a(h_b + h_c)} \geq \sqrt{\frac{R}{2r}} + \sqrt{\frac{R}{2r}} + \sqrt{\frac{R}{2r}} \overset{Panaitopol}{\geq} \sum_{cyc} \sqrt{\frac{m_a}{h_a}} \text{ and so,}$$

$$\left(\frac{R}{2r}\right)^{\frac{3}{2}} \cdot \sum_{cyc} \frac{r_a^2 + r_b r_c}{w_a(h_b + h_c)} \geq \sum_{cyc} \sqrt{\frac{m_a}{h_a}} \quad \forall \Delta ABC, '' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

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4323. In ΔABC the following relationship holds:

$$ar_b + br_c + cr_c \geq 6F$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$ar_b + br_c + cr_c \overset{AM-GM}{\geq} 3\sqrt[3]{abcr_ar_br_c} = 3\sqrt[3]{abc\cdot\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}} =$$

$$= 3\sqrt[3]{abc\cdot\frac{F^3s}{s(s-a)(s-b)(s-c)}} = 3\sqrt[3]{4RF\cdot\frac{F^3s}{F^2}} = 3\sqrt[3]{4RsF^2} = 3\sqrt[3]{4Rs\cdot r^2s^2} =$$

$$= 3s\sqrt[3]{4Rr^2} \overset{EULER}{\geq} 3s\cdot\sqrt[3]{8r\cdot r^2} = 3s\cdot\sqrt[3]{8r^3} = 3s\cdot\sqrt[3]{(2r)^3} = 3s\cdot 2r =$$

$$= 6rs = 6F$$

Equality holds for $a = b = c$.

4324. In any Δ ABC the following relationship holds :

$$\left(\frac{R}{2r}\right)^3\cdot\sum_{cyc}\frac{r_a^3 + r_ar_br_c}{w_a\left(h_b^2 + h_c^2\right)} \geq \sum_{cyc}\left(\sqrt{\frac{m_a}{h_a}}\cdot\frac{r_br_c}{w_bw_c}\right)$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$(b+c)^2 \overset{?}{\geq} 32Rr\cdot\cos^2\frac{A}{2} = 8r(r_b + r_c) = 8r^2s\left(\frac{1}{s-b} + \frac{1}{s-c}\right)$$

$$= 8(s-a)(s-b)(s-c)\cdot\frac{a}{(s-b)(s-c)} = 4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \overset{?}{\geq} 0 \Leftrightarrow (b+c-2a)^2 \overset{?}{\geq} 0 \to \text{true}$$

$$\therefore b + c \geq \sqrt{32Rr}\cdot\cos\frac{A}{2} \text{ and analogs} \Rightarrow \frac{w_a}{h_a} = \frac{2R\cdot 2bc\cdot\cos\frac{A}{2}}{(b+c)bc} \leq \frac{4R\cdot\cos\frac{A}{2}}{\sqrt{32Rr}\cdot\cos\frac{A}{2}} = \sqrt{\frac{R}{2r}}$$

$$\Rightarrow w_a\left(h_b^2 + h_c^2\right) \leq \sqrt{\frac{R}{2r}}\cdot h_a\left(h_b^2 + h_c^2\right) \leq \sqrt{\frac{R}{2r}}\cdot h_a\left(w_b^2 + w_c^2\right) \overset{Lascu + AM-GM}{\leq}$$

$$\sqrt{\frac{R}{2r}}\cdot h_a\cdot\left(s(s-b) + s(s-c)\right) = \sqrt{\frac{R}{2r}}\cdot\frac{2rs}{a}\cdot sa \Rightarrow w_a\left(h_b^2 + h_c^2\right) \leq \sqrt{\frac{R}{2r}}\cdot 2s^2r$$

$$\text{and analogs} \Rightarrow \left(\frac{R}{2r}\right)^3 \cdot \sum_{cyc} \frac{r_a^3 + r_a r_b r_c}{w_a(h_b^2 + h_c^2)} \overset{①}{\geq} \frac{\left(\frac{R}{2r}\right)^3}{\sqrt{\frac{R}{2r}} \cdot 2s^2 r} \cdot \left(\sum_{cyc} r_a^3 + 3r_a r_b r_c\right)$$

$$\text{Again}, \sum_{cyc}\left(\sqrt{\frac{m_a}{h_a}} \cdot \frac{r_b r_c}{w_b w_c}\right) \overset{Panaitopol}{\leq} \sqrt{\frac{R}{2r}} \cdot \sum_{cyc} \frac{s(s-a)}{h_b h_c} =$$

$$\sqrt{\frac{R}{2r}} \cdot \frac{s}{4r^2 s^2} \cdot (s(s^2 + 4Rr + r^2) - 12Rrs)$$

$$\therefore \sum_{cyc}\left(\sqrt{\frac{m_a}{h_a}} \cdot \frac{r_b r_c}{w_b w_c}\right) \overset{②}{\leq} \sqrt{\frac{R}{2r}} \cdot \frac{s^2 - 8Rr + r^2}{4r^2} \therefore ① \text{ and } ② \Rightarrow \text{it suffices to prove :}$$

$$\frac{\left(\frac{R}{2r}\right)^3}{2s^2 r} \cdot ((4R + r)^3 - (12R - 3r)s^2) \overset{?}{\geq} \frac{R}{2r} \cdot \frac{s^2 - 8Rr + r^2}{4r^2}$$

$$\Leftrightarrow R^2((4R + r)^3 - (12R - 3r)s^2) \overset{?}{\geq} 2s^2 r(s^2 - 8Rr + r^2)$$

$$\text{and} \because (12R - 3r)s^2 \overset{Gerretsen}{\leq} (12R - 3r)(4R^2 + 4Rr + 3r^2) \text{ and } s^2(s^2 - 8Rr + r^2) \overset{Gerretsen}{\leq} (4R^2 + 4Rr + 3r^2)(4R^2 - 4Rr + 4r^2) \therefore \text{it suffices to prove :}$$

$$R^2\left((4R + r)^3 - (12R - 3r)(4R^2 - 4Rr + 4r^2)\right) \overset{?}{\geq}$$

$$2r(4R^2 - 4Rr + 4r^2)(4R^2 - 4Rr + 4r^2) \Leftrightarrow 8t^5 - 10t^4 - 6t^3 - 7t^2 - 4t - 12 \overset{?}{\geq} 0$$

$$\left(t = \frac{R}{r}\right) \Leftrightarrow (t - 2)(8t^4 + 6t^3 + 6t^2 + 5t + 6) \overset{?}{\geq} 0 \to \text{true} \because t \overset{Euler}{\geq} 2 \text{ and so,}$$

$$\left(\frac{R}{2r}\right)^3 \cdot \sum_{cyc} \frac{r_a^3 + r_a r_b r_c}{w_a(h_b^2 + h_c^2)} \geq \sum_{cyc}\left(\sqrt{\frac{m_a}{h_a}} \cdot \frac{r_b r_c}{w_b w_c}\right) \forall \Delta ABC,$$

$''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$

4325. In ΔABC the following relationship holds:

$$r_a r_b r_c \geq \sqrt[4]{27} \cdot F\sqrt{F}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$r_a r_b r_c = \frac{F}{s-a} \cdot \frac{F}{s-b} \cdot \frac{F}{s-c} = \frac{F^3 s}{s(s-a)(s-b)(s-c)} =$$

$$= \frac{F^3 s}{F^2} = Fs = F\sqrt{s^2} = F\sqrt{s \cdot s} \geq$$

$$\overset{MITRINOVIC}{\geq} F\sqrt{s\cdot 3\sqrt{3}r} = F\sqrt{F\cdot\sqrt{27}} = \sqrt[4]{27}F\sqrt{F}$$

Equality holds for $a=b=c$.

4326. *In any ΔABC following relationship holds:*

$$\frac{R}{r}\sum_{cyc}\sqrt{\frac{m_a^3}{w_a^2h_a}}+\frac{1}{3r^2}\sum r_ar_b\geq\frac{7}{2}\sum\frac{r_a}{h_a}+\frac{9}{2}\sqrt[3]{1-\sum\left(\frac{m_a-w_a}{m_a+w_a}\right)^2}$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Tapas Das-India

$$\frac{R}{r}\sum_{cyc}\sqrt{\frac{m_a^3}{w_a^2h_a}}\overset{m_a\geq w_a\geq h_a}{\geq}\frac{3R}{r}=3x,\frac{R}{r}=x\geq 2\ (Euler)$$

$$\frac{1}{3r^2}\sum r_ar_b=\frac{1}{3}\left(\frac{s^2}{r^2}\right)\overset{Gerretsn}{\geq}\frac{1}{3}\frac{16Rr-5r^2}{r^2}=\frac{1}{3}(16x-5)$$

$$\frac{7}{2}\sum\frac{r_a}{h_a}=\frac{7}{2}\cdot\frac{1}{2}\sum\frac{a}{s-a}=\frac{7}{2}\left(\frac{2R}{r}-1\right)=\frac{7}{2}(2x-1)$$

$$\frac{9}{2}\sqrt[3]{1-\sum\left(\frac{m_a-w_a}{m_a+w_a}\right)^2}\geq\frac{9}{2}\ as\ m_a\geq w_a\ then\ m_a-w_a\geq 0$$

We need to show:

$$\frac{R}{r}\sum_{cyc}\sqrt{\frac{m_a^3}{w_a^2h_a}}+\frac{1}{3r^2}\sum r_ar_b\geq\frac{7}{2}\sum\frac{r_a}{h_a}+\frac{9}{2}\sqrt[3]{1-\sum\left(\frac{m_a-w_a}{m_a+w_a}\right)^2}$$

$$3x+\frac{1}{3}(16x-5)\geq\frac{7}{2}(2x-1)+\frac{9}{2}$$

$$18x+32x-10\geq 42x\text{-}21+27\ or\ 8x\geq 16\ or\ x\geq 2\ True$$

Equality holds for an equilateral triangle.

4327. In ΔABC the following relationship holds:

$$h_ah_bh_c\leq\sqrt[4]{27}\cdot F\sqrt{F}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$h_ah_bh_c=\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}=\frac{8F^3}{abc}=\frac{8F^3}{4RF}=\frac{2F^2}{R}\leq\frac{2F^2}{2r}=\frac{F^2}{r}=\frac{F\cdot F}{2}=\frac{F\cdot rs}{r}=$$

$$= Fs = F\sqrt{s^2} = F\sqrt{s\cdot s} \overset{MITRINOVIC}{\geq} F\sqrt{s\cdot 3\sqrt{3}r} = F\sqrt{F\cdot\sqrt{27}} = \sqrt[4]{27}F\sqrt{F}$$

Equality holds for $a = b = c$.

4328. *In ΔABC following relationship holds:*

$$\left(\frac{R}{r}\right)^2\sum\sqrt{\frac{m_a^5}{w_a^3h_a^2}}+\frac{1}{3r^2}\sum r_ar_b \geq 5\sum\frac{r_a}{h_a}+2\sqrt[3]{\frac{32R}{r}-37}+16\sum\left(\frac{m_a-w_a}{m_a+h_a}\right)^2$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Tapas Das-India

$$\sum\sqrt{\frac{m_a^5}{w_a^3h_a^2}} \overset{m_a\geq w_a\geq h_a}{\geq} \sum 1 = 3\ (1)$$

$$\frac{1}{3r^2}\sum r_ar_b = \frac{1}{3}\frac{s^2}{r^2} \overset{Gerretsen}{\geq} \frac{16Rr-5r^2}{3r^2} \overset{\frac{R}{r}=x\geq 2}{=} \frac{1}{3}(16x-5)\ (2)$$

$$5\sum\frac{r_a}{h_a} = 5\sum\frac{\frac{F}{s-a}}{\frac{2F}{a}} = \frac{5}{2}\sum\frac{a}{s-a} = \frac{5}{2}\left(\frac{4R}{r}-2\right) = 5(2x-1)\ (3)$$

$$16\sum\left(\frac{m_a-w_a}{m_a+h_a}\right)^2 \overset{w_a\geq h_a}{\geq} 16\sum\left(\frac{m_a-h_a}{m_a+h_a}\right)^2 =$$

$$= 16\sum\left(\frac{\frac{m_a}{h_a}-1}{\frac{m_a}{h_a}+1}\right)^2 \overset{\substack{Panaitopol\\ m_a\geq h_a}}{\geq} 16\sum\left(\frac{\frac{R}{2r}-1}{1+1}\right)^2 =$$

$$= \sum\left(\frac{R}{r}-2\right)^2 = \sum(x-2)^2 = 3(x-2)^2\ (4)$$

We need to show:

$$\left(\frac{R}{r}\right)^2\sum\sqrt{\frac{m_a^5}{w_a^3h_a^2}}+\frac{1}{3r^2}\sum r_ar_b \geq 5\sum\frac{r_a}{h_a}+2\sqrt[3]{\frac{32R}{r}-37}+16\sum\left(\frac{m_a-w_a}{m_a+h_a}\right)^2$$

Using $(1),(2),(3),(4)$ we get:

$$3x^2+\frac{1}{3}(16x-5) \geq 5(2x-1)+2\sqrt[3]{32x-37}+3(x-2)^2$$

$$9x^2+16x-5 \geq 9x^2-6x+21+6\sqrt[3]{32x-37} \text{ or } 11x-13 \geq 3\sqrt[3]{32x-37}\ (A)$$

$$\text{or let } y = \sqrt[3]{32x-37} \text{ then } y \overset{x\geq 2}{\geq} \sqrt[3]{32\times 2-37} = \sqrt[3]{27} = 3 \ \&\ x = \frac{y^3+37}{32}$$

$$\text{From } (A) \text{ we have } 11\left(\frac{y^3+37}{32}\right)-13 \geq 3y \text{ or } 11y^3-96y-9 \geq 0$$

$$\text{or } (y-3)(11y^2+33y+3) \geq 0 \text{ true as } y \geq 3$$

Equality holds for an equilateral triangle.

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4329. In any ΔABC the following relationship holds :

$$\frac{R}{r}\cdot\sum_{cyc}\sqrt{\frac{m_a}{h_a}}+\sqrt[4]{\frac{1}{r^2}\sum_{cyc}r_ar_b}\geq\frac{3}{2}\cdot\left(1+\sum_{cyc}\frac{r_a}{h_a}\right)+\sqrt[4]{16\cdot\frac{R}{r}-5}+\sum_{cyc}\left(\frac{m_a-h_a}{m_a}\right)^2$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{3}{2}\cdot\left(1+\sum_{cyc}\frac{r_a}{h_a}\right)=\frac{3}{2}\cdot\left(1+\sum_{cyc}\frac{s\tan\frac{A}{2}\cdot 4R\cdot\cos\frac{A}{2}\sin\frac{A}{2}}{2rs}\right)$$

$$=\frac{3}{2}\cdot\left(1+\frac{2R}{r}\cdot\sum_{cyc}\sin^2\frac{A}{2}\right)=\frac{3}{2}\cdot\left(1+\frac{2R}{r}\cdot\frac{2R-r}{2R}\right)=\frac{3R}{r}$$

$$\therefore\frac{R}{r}\cdot\sum_{cyc}\sqrt{\frac{m_a}{h_a}}-\frac{3}{2}\cdot\left(1+\sum_{cyc}\frac{r_a}{h_a}\right)=\frac{R}{r}\cdot\sum_{cyc}\left(\sqrt{\frac{m_a}{h_a}}-1\right)=\frac{R}{r}\cdot\sum_{cyc}\frac{m_a-h_a}{\sqrt{h_a}\cdot\left(\sqrt{m_a}+\sqrt{h_a}\right)}$$

$$\geq 2\cdot\sum_{cyc}\frac{m_a-h_a}{m_a+h_a}\left(\because h_a\leq m_a \text{ and analogs and } \frac{R}{r}\overset{\text{Euler}}{\geq}2\right)$$

$$\therefore\frac{R}{r}\cdot\sum_{cyc}\sqrt{\frac{m_a}{h_a}}-\frac{3}{2}\cdot\left(1+\sum_{cyc}\frac{r_a}{h_a}\right)\geq 2\cdot\sum_{cyc}\frac{m_a-h_a}{m_a+h_a}\rightarrow(1)$$

$$\text{Now, } 2\cdot\frac{m_a-h_a}{m_a+h_a}\overset{?}{\geq}\left(\frac{m_a-h_a}{m_a}\right)^2\Leftrightarrow 2m_a^2\overset{?}{\geq}(m_a-h_a)(m_a+h_a)\ (\because m_a-h_a\geq 0)$$

$$\Leftrightarrow 2m_a^2\overset{?}{\geq}m_a^2-h_a^2\Leftrightarrow m_a^2+h_a^2\overset{?}{\geq}0\rightarrow\text{true (strict inequality)}$$

$$\therefore 2\cdot\frac{m_a-h_a}{m_a+h_a}\geq\left(\frac{m_a-h_a}{m_a}\right)^2\ \&\ \text{analogs}\Rightarrow 2\cdot\sum_{cyc}\frac{m_a-h_a}{m_a+h_a}\geq\sum_{cyc}\left(\frac{m_a-h_a}{m_a}\right)^2\rightarrow(2)$$

$$\therefore(1)\text{ and }(2)\Rightarrow\frac{R}{r}\cdot\sum_{cyc}\sqrt{\frac{m_a}{h_a}}\overset{(*)}{\geq}\frac{3}{2}\cdot\left(1+\sum_{cyc}\frac{r_a}{h_a}\right)+\sum_{cyc}\left(\frac{m_a-h_a}{m_a}\right)^2$$

$$\text{Again, }\sqrt[4]{\frac{1}{r^2}\sum_{cyc}r_ar_b}=\sqrt[4]{\frac{s^2}{r^2}}\overset{\text{Gerretsen}}{\geq}\sqrt[4]{\frac{16Rr-5r^2}{r^2}}\therefore\sqrt[4]{\frac{1}{r^2}\sum_{cyc}r_ar_b}\overset{(**)}{\geq}\sqrt[4]{16\cdot\frac{R}{r}-5}$$

$$\text{So, }(*)+(**)\Rightarrow\frac{R}{r}\cdot\sum_{cyc}\sqrt{\frac{m_a}{h_a}}+\sqrt[4]{\frac{1}{r^2}\sum_{cyc}r_ar_b}\geq$$

$$\frac{3}{2}\cdot\left(1+\sum_{cyc}\frac{r_a}{h_a}\right)+\sqrt[4]{16.\frac{R}{r}-5+\sum_{cyc}\left(\frac{m_a-h_a}{m_a}\right)^2}\ \forall\ \Delta\ ABC,$$

$''=''$ iff Δ ABC is equi*later*al (QED)

4330. In ΔABC the following relationship holds:

$$3\sqrt{3}r^2 \le F \le \frac{3\sqrt{3}}{4}R^2$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$F = rs \overset{MITRINOVIC}{\geq} r\cdot 3\sqrt{3}r = 3\sqrt{3}r^2$$

$$F = rs \overset{EULER}{\leq} \frac{R}{2}\cdot s \overset{MITRINOVIC}{\leq} \frac{R}{2}\cdot\frac{3\sqrt{3}}{2}R = \frac{3\sqrt{3}}{4}R^2$$

Equality holds for an equilateral triangle.

4331. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\left(2r_a^2+r_b^2+r_c^2\right)^2}{\left(m_b^2+m_c^2\right)(r_br_c+r_ar)} \geq \frac{2R^2}{r^2}+\frac{r_a+r_b+r_c}{r}+1+\frac{\sum_{cyc}(a-b)^2}{2r^2}+\frac{\sum_{cyc}(m_a-m_b)^2}{6r^2}$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Nawar Alasadi-Iraq

$$\begin{aligned} r_br_c+r_ar &= s(s-a)+(s-b)(s-c)\\ &= 2s^2-s(a+b+c)+bc = 2s^2-2s^2+bc = bc \end{aligned}$$

By the Cauchy-Schwarz inequality:

$$\sum_{cyc}\frac{\left(2r_a^2+r_b^2+r_c^2\right)^2}{\left(m_b^2+m_c^2\right)bc} \geq \frac{\left(\sum_{cyc}\left(2r_a^2+r_b^2+r_c^2\right)\right)^2}{\sum_{cyc}b\,c\left(m_b^2+m_c^2\right)} = \frac{16\left(r_a^2+r_b^2+r_c^2\right)^2}{\sum_{cyc}b\,c\left(m_b^2+m_c^2\right)}$$

We evaluate the denominator:

$$m_b^2+m_c^2 = \frac{4a^2+b^2+c^2}{4}$$

$$\sum_{cyc} b\,c(m_b^2+m_c^2) = \frac{1}{4}\sum_{cyc} b\,c(4a^2+b^2+c^2) = abc(a+b+c)+\frac{1}{4}\sum_{cyc} b\,c(b^2+c^2)$$

$$= 8Rrs^2+\frac{1}{4}[2(s^2+r^2+4Rr)(s^2-r^2-4Rr)-8Rrs^2]$$

$$= \frac{1}{2}s^4+6Rrs^2-\frac{1}{2}(r^2+4Rr)^2$$

The Right Hand Side (RHS) simplifies to:

$$\text{RHS} = \frac{2R^2}{r^2}+\frac{4R+r}{r}+1+\frac{2s^2-24Rr-6r^2}{2r^2}+\frac{2\sum_{cyc} m_a^2-2\sum_{cyc} m_a m_b}{6r^2}$$

$$= \frac{2R^2}{r^2}+\frac{4R}{r}+2+\frac{s^2-12Rr-3r^2}{r^2}+\frac{\sum_{cyc} m_a^2-\sum_{cyc} m_a m_b}{3r^2}$$

Using the fundamental triangle inequalities and

$\sum_{cyc} r_a^2 = (4R+r)^2-2s^2$, **we obtain:**

$$\frac{16\left(r_a^2+r_b^2+r_c^2\right)^2}{\frac{1}{2}s^4+6Rrs^2-\frac{1}{2}(r^2+4Rr)^2} \geq \text{RHS}$$

4332. In ΔABC the following relationship holds:

$$\sum_{cyc} a(m_a-m_b) \leq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$WLOG: \ a \leq b \leq c \rightarrow m_a \geq m_b \geq m_c$$

$$These\ conditions\ are\ suitable\ for\ the\ rearrangements'\ inequality:$$

$$B_{min} = am_a+bm_b+cm_c$$

$$C_{mixed} = am_b+bm_c+cm_a$$

$$B_{min} \leq C_{mixed}$$

$$am_a+bm_b+cm_c \leq am_b+bm_c+cm_a$$

$$\sum_{cyc} a(m_a-m_b) \leq 0$$

$$Equality\ holds\ if\ a=b=c$$

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4333. In any Δ ABC the following relationship holds :

$$\frac{r_a^4h_b^2h_c^2}{h_a^4(r_b^2h_c^2+r_c^2h_b^2)}+\frac{r_b^4h_c^2h_a^2}{h_b^4(r_c^2h_a^2+r_a^2h_c^2)}+\frac{r_c^4h_a^2h_b^2}{h_c^4(r_a^2h_b^2+r_b^2h_a^2)}\geq\frac{(2R-r)^2}{6r^2}$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{r_a^4h_b^2h_c^2}{h_a^4(r_b^2h_c^2+r_c^2h_b^2)}=\sum_{cyc}\frac{\left(\frac{r_a^2}{h_a^2}\right)^2}{\frac{r_b^2}{h_b^2}+\frac{r_c^2}{h_c^2}}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}\frac{r_a^2}{h_a^2}\right)^2}{2\sum_{cyc}\frac{r_a^2}{h_a^2}}=\frac{1}{2}\cdot\sum_{cyc}\frac{r_a^2}{h_a^2}$$

$$\geq\frac{1}{6}\cdot\left(\sum_{cyc}\frac{r_a}{h_a}\right)^2=\frac{1}{6}\cdot\left(\sum_{cyc}\frac{s\tan\frac{A}{2}\cdot 4R\cdot\cos\frac{A}{2}\sin\frac{A}{2}}{2rs}\right)^2=\frac{1}{6}\cdot\left(\frac{2R}{r}\cdot\sum_{cyc}\sin^2\frac{A}{2}\right)^2$$

$$=\frac{1}{6}\cdot\left(\frac{2R}{r}\cdot\frac{2R-r}{2R}\right)^2=\frac{(2R-r)^2}{6r^2}\ \text{and so,}$$

$$\frac{r_a^4h_b^2h_c^2}{h_a^4(r_b^2h_c^2+r_c^2h_b^2)}+\frac{r_b^4h_c^2h_a^2}{h_b^4(r_c^2h_a^2+r_a^2h_c^2)}+\frac{r_c^4h_a^2h_b^2}{h_c^4(r_a^2h_b^2+r_b^2h_a^2)}\geq\frac{(2R-r)^2}{6r^2}$$

$\forall\ \Delta$ ABC, $''='' $ iff Δ ABC is equilateral (QED)

4334. In acute ΔABC the following relationship holds:

$$\cot\frac{A}{2}\csc A+\cot\frac{B}{2}\csc B+\cot\frac{C}{2}\csc C\geq 6$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\cot\frac{A}{2}\csc A+\cot\frac{B}{2}\csc B+\cot\frac{C}{2}\csc C\overset{AM-GM}{\geq}3\cdot\sqrt[3]{\frac{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}{\sin A\sin B\sin C}}$$

$$=3\cdot\sqrt[3]{\frac{\frac{s}{r}}{\frac{a}{2R}\cdot\frac{b}{2R}\cdot\frac{c}{2R}}}=3\cdot 2R\cdot\sqrt[3]{\frac{s}{abcr}}=6R\cdot\sqrt[3]{\frac{s}{4Rrsr}}=6R\sqrt[3]{\frac{1}{4Rr^2}}\overset{EULER}{\geq}$$

$$\geq 6R \cdot \sqrt[3]{\frac{1}{4R \cdot \frac{R^2}{4}}} = 6R \cdot \frac{1}{R} = 6$$

Equality holds for $a = b = c$.

4335. In ΔABC the following relationship holds:

$$\left(\sum_{cyc} \frac{a^2}{b^2 + c^2}\right)\left(\sum_{cyc} \frac{rr_a}{bc}\right) \leq \frac{(2R - r)^2}{4Rr}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc} \frac{a^2}{b^2 + c^2} \overset{AM-GM}{\leq} \frac{1}{2}\sum_{cyc} \frac{a^2}{bc} = \frac{1}{2abc}\sum_{cyc} a^3 = \frac{1}{8Rpr} \cdot 2(p^3 - 3pr^2 - 6Rpr) =$$

$$= \frac{1}{4Rr}(p^2 - 3r^2 - 6Rr) \overset{Gerretsen}{\leq} \frac{1}{4Rr}(4R^2 + 4Rr + 3r^2 - 3r^2 - 6Rr) = \frac{2R - r}{2r} \quad (1)$$

$$\sum_{cyc} \frac{rr_a}{bc} = r\sum_{cyc} \frac{r_a}{bc} = \frac{r}{abc}(ar_a + br_b + cr_c) = \frac{r}{4Rrp} \cdot 2p(2R - r) = \frac{2R - r}{2R} \quad (2)$$

From (1) and (2):

$$\left(\sum_{cyc} \frac{a^2}{b^2 + c^2}\right)\left(\sum_{cyc} \frac{rr_a}{bc}\right) \leq \frac{2R - r}{2r} \cdot \frac{2R - r}{2R} = \frac{(2R - r)^2}{4Rr}$$

Equality holds for: $a = b = c$

4336. In ΔABC the following relationship holds:

$$\sum_{cyc} m_a(\sin A - \sin B) \leq 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{WLOG: } a \leq b \leq c \Rightarrow \begin{cases} m_a \geq m_b \geq m_c \\ \sin A \leq \sin B \leq \sin C \end{cases}$$

By rearrangements' inequality:

$$m_a \sin A + m_b \sin B + m_c \sin C \leq m_a \sin B + m_b \sin C + m_c \sin A$$

$$m_a(\sin A - \sin B) + m_b(\sin B - \sin C) + m_c(\sin C - \sin B) \leq 0$$

$$\sum_{cyc} m_a(\sin A - \sin B) \leq 0$$

Equality holds for $a = b = c$.

4337. In ΔABC the following relationship holds:

$$\frac{r_a^3}{r_b^2 - r_b r_c + r_c^2} + \frac{r_b^3}{r_c^2 - r_a r_c + r_a^2} + \frac{r_c^3}{r_a^2 - r_b r_a + r_b^2} \geq 9r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{r_a^3}{r_b^2 - r_b r_c + r_c^2} + \frac{r_b^3}{r_c^2 - r_a r_c + r_a^2} + \frac{r_c^3}{r_a^2 - r_b r_a + r_b^2} =$$
$$= \frac{r_a^3(r_b + r_c)}{r_b^3 + r_c^3} + \frac{r_b^3(r_a + r_c)}{r_a^3 + r_c^3} + \frac{r_c^3(r_b + r_a)}{r_b^3 + r_a^3}$$

The last expression satisfies the conditions of the Walter Janous inequality

$$x, y, z, a, b, c > 0$$
$$\frac{x}{y+z}(b+c) + \frac{y}{x+z}(a+c) + \frac{z}{y+x}(b+a) \geq \sqrt{3(ab+bc+ac)}$$

Here $x = r_a^3, y = r_b^3, z = r_c^3, a = r_a, b = r_b, c = r_c$. *Then*:

$$\frac{r_a^3(r_b + r_c)}{r_b^3 + r_c^3} + \frac{r_b^3(r_a + r_c)}{r_a^3 + r_c^3} + \frac{r_c^3(r_b + r_a)}{r_b^3 + r_a^3} \geq$$
$$\geq \sqrt{3(r_a r_b + r_c r_b + r_a r_c)} = \sqrt{3.s^2} = \sqrt{3}s \geq \sqrt{3}.3\sqrt{3}r = 9r$$

Equality holds for: $a = b = c$.

4338. In ΔABC the following relationship holds:

$$\sum_{cyc} h_a(m_a - m_b) \geq 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

WLOG: $a \leq b \leq c \Rightarrow \begin{cases} h_a \geq h_b \geq h_c \\ m_a \geq m_b \geq m_c \end{cases}$

By rearrangements' inequality

$$h_a m_a + h_b m_b + h_c m_c \geq h_a m_b + h_b m_c + h_c m_a$$
$$h_a(m_a - m_b) + h_b(m_b - m_c) + h_c(m_c - m_a) \geq 0$$

$$\sum_{cyc} h_a\,(m_a - m_b) \geq 0$$

Equality holds for $a = b = c$.

Observation 1

$$a \leq b \Leftrightarrow \frac{1}{a} \geq \frac{1}{b} \Leftrightarrow \frac{2F}{a} \geq \frac{2F}{b} \Leftrightarrow h_a \geq h_b$$

Observation 2

$$a \leq b \Leftrightarrow m_a \geq m_b$$

$$m_a \geq m_b \Leftrightarrow m_a^2 \geq m_b^2 \Leftrightarrow \frac{1}{2}(b^2+c^2) - \frac{1}{4}a^2 \geq \frac{1}{2}(a^2+c^2) - \frac{1}{4}b^2$$

$$\Leftrightarrow 2(b^2+c^2) - a^2 \geq 2(a^2+c^2) - b^2$$

$$\Leftrightarrow 2b^2 + 2c^2 - a^2 \geq 2a^2 + 2c^2 - b^2$$

$$\Leftrightarrow 3a^2 \leq 3b^2 \Leftrightarrow a^2 \leq b^2 \Leftrightarrow a \leq b$$

4339. In any ΔABC the following relationship holds :

$$\frac{\csc^3\frac{A}{2}}{\csc^2\frac{B}{2} - \csc\frac{B}{2}\csc\frac{C}{2} + \csc^2\frac{C}{2}} + \frac{\csc^3\frac{B}{2}}{\csc^2\frac{C}{2} - \csc\frac{C}{2}\csc\frac{A}{2} + \csc^2\frac{A}{2}} + \frac{\csc^3\frac{C}{2}}{\csc^2\frac{A}{2} - \csc\frac{A}{2}\csc\frac{B}{2} + \csc^2\frac{B}{2}} \geq 6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

We shall first prove :

$$s\left(r^2((4R+r)^2 - 2s^2)\right)^2 \overset{?}{\geq} (*)\; 9s^2r^4(s(4Rr+r^2) - 6r^2s)$$

$$\Leftrightarrow 4s^4 - (64R^2 + 68Rr - 41r^2)s^2 + (4R+r)^4 \overset{?}{\geq} (1)\; 0$$

and $\because (s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove (1),

it suffices to prove : LHS of (1) $\overset{?}{\geq} 4(s^2 - 16Rr + 5r^2)^2$

$$\Leftrightarrow 256R^4 + 256R^3r - 928R^2r^2 + 656Rr^3 - 99r^4 \overset{?}{\geq} (2)\; (64R^2 - 60Rr - r^2)s^2$$

Finally, RHS of (2) $\overset{\text{Gerretsen}}{\leq} (64R^2 - 60Rr - r^2)(4R^2 + 4Rr + 3r^2) \overset{?}{\leq}$ LHS of (2)

$$\Leftrightarrow 20t^3-73t^2+70t-8\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow (t-2)(20t^2-33t+4)\overset{?}{\geq}0$$

$$\rightarrow \text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow (2)\Rightarrow (1)\Rightarrow \boxed{(*)\text{ is true}}\ \textit{and now,}$$

$$\sum_{cyc}\frac{\csc^3\frac{A}{2}}{\csc^2\frac{B}{2}-\csc\frac{B}{2}\csc\frac{C}{2}+\csc^2\frac{C}{2}}=\sum_{cyc}\frac{\frac{1}{\sin^3\frac{A}{2}}}{\frac{1}{\sin^2\frac{B}{2}}-\frac{1}{\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{1}{\sin^2\frac{C}{2}}}$$

$$=\sum_{cyc}\frac{\left(\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}\right)^2}{\sin\frac{A}{2}\left(\sin^2\frac{B}{2}-\sin\frac{B}{2}\sin\frac{C}{2}+\sin^2\frac{C}{2}\right)}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}\right)^2}{\sum_{cyc}\left(\sin\frac{A}{2}\left(\sin^2\frac{B}{2}-\sin\frac{B}{2}\sin\frac{C}{2}+\sin^2\frac{C}{2}\right)\right)}$$

$$\geq\frac{\frac{2}{3}\left(\sum_{cyc}\sin\frac{A}{2}\right)\left(\sum_{cyc}\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}\right)^2}{\left(\sum_{cyc}\sin\frac{A}{2}\right)\left(\sum_{cyc}\sin\frac{A}{2}\sin\frac{B}{2}\right)-6\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\left(\because 1\overset{\text{Jensen}}{\geq}\frac{2}{3}\left(\sum_{cyc}\sin\frac{A}{2}\right)\right)\overset{?}{\geq}6$$

$$\Leftrightarrow\frac{\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2y^2\right)^2}{x^2y^2z^2\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-6xyz\right)}\overset{?}{\geq}9\ \left(x=\sin\frac{A}{2},y=\sin\frac{B}{2},z=\sin\frac{C}{2}\right)\ (\bullet)\ AJV$$

Assigning $y+z=X, z+x=Y, x+y=Z\Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ *and* $Z+X-Y=2y>0\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y\Rightarrow X,Y,Z$ form sides of *a* tri*a*ngle with semiperimeter, circumr*a*dius *a*nd inr*a*dius $=s',R,r'$ *(say)*

and then : $\sum_{cyc}x=s', xyz=r'^2s'\sum_{cyc}xy=4R'r'+r'^2$ *and*

$$\sum_{cyc}x^2y^2=r'^2((4R'+r')^2-2s'^2)\ \textit{and so},(\bullet)\Leftrightarrow$$

$$\frac{s'\left(r'^2\left((4R'+r')^2-2s'^2\right)\right)^2}{s'^2.r'^4(s'(4R'r'+r'^2)-6r'^2.s')}\overset{?}{\geq}9\rightarrow\text{true via }(*)\Rightarrow\boxed{(\bullet)\text{ is true}}\ \textit{and so,}$$

$$\frac{\csc^3\frac{A}{2}}{\csc^2\frac{B}{2}-\csc\frac{B}{2}\csc\frac{C}{2}+\csc^2\frac{C}{2}}+\frac{\csc^3\frac{B}{2}}{\csc^2\frac{C}{2}-\csc\frac{C}{2}\csc\frac{A}{2}+\csc^2\frac{A}{2}}+$$

$$\frac{\csc^3\frac{C}{2}}{\csc^2\frac{A}{2}-\csc\frac{A}{2}\csc\frac{B}{2}+\csc^2\frac{B}{2}}\geq 6\ \forall\ ABC,''='' \text{ iff } \Delta ABC \text{ is equi}\textit{lateral}\ (QED)$$

4340. In ΔABC the following relationship holds:

$$\sum_{cyc} a(h_a - h_b) \leq 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

WLOG: $a \leq b \leq c \Rightarrow h_a \geq h_b \geq h_c$.

By rearrangements' inequality:

$$ah_a + bh_b + ch_c \leq ah_b + bh_c + ch_a$$

$$a(h_a - h_b) + b(h_b - h_c) + c(h_c - h_a) \leq 0$$

$$\sum_{cyc} a(h_a - h_b) \leq 0$$

Equality holds for $a = b = c$.

Observation:

$$a \leq b \Rightarrow h_a \geq h_b$$

$$h_a \geq h_b \Leftrightarrow \frac{2F}{a} \geq \frac{2F}{b} \Leftrightarrow \frac{1}{a} \geq \frac{1}{b} \Leftrightarrow a \leq b$$

4341.

In all ΔABC such that $a \neq b \neq c$ the following relationship holds:

$$\frac{a^2 r_a}{(a-b)(a-c)} + \frac{b^2 r_b}{(b-a)(b-c)} + \frac{c^2 r_c}{(c-b)(c-a)} < \frac{27R^2}{4r}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{a^2 F}{(S-a)(a-b)(a-c)} + \frac{b^2 F}{(S-b)(b-c)(b-a)} + \frac{c^2 F}{(S-c)(c-a)(c-b)} =$$

$$= F\left(\frac{a^2}{(S-a)(a-b)(a-c)} + \frac{b^2}{(S-b)(b-c)(b-a)} + \frac{c^2}{(S-c)(c-a)(c-b)}\right)$$

Let's prove that:

$$\frac{a^2}{(s-a)(a-b)(a-c)} + \frac{b^2}{(s-b)(b-c)(b-a)} + \frac{c^2}{(s-c)(c-a)(c-b)} =$$

$$= \frac{s^2}{(s-a)(s-b)(s-c)}$$

So that for three distinct points a,b,c the Lagrange interpolation formula (expansion) for the function $f(x)=x^2$ is as follows:

$$x^2=a^2.\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^2.\frac{(x-a)(x-c)}{(b-a)(b-c)}+c^2.\frac{(x-a)(x-b)}{(c-b)(c-a)}$$

Showing that the expression is identically true. Let's find the derivative of both sides of the expression:

$$2x=a^2.\frac{(2x-(b+c))}{(a-b)(a-c)}+b^2.\frac{(2x-(a+c))}{(b-a)(b-c)}+c^2.\frac{(2x-(a+b))}{(c-b)(c-a)}$$

second derivative:

$$2=\frac{2a^2}{(a-b)(a-c)}+\frac{2b^2}{(b-a)(b-c)}+\frac{2c^2}{(c-b)(c-a)}$$

Now let's prove that:

$$1=\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-a)(b-c)}+\frac{c^2}{(c-b)(c-a)}$$

$$LHS=\frac{a^2(b-c)-b^2(a-c)-c^2(a-b)}{(a-b)(a-c)(b-c)}$$

$$Numerator=a^2(b-c)-b^2(a-c)-c^2(a-b)=$$

$$a^2(b-c)-b^2a+b^2c+c^2a-c^2b=a^2(b-c)-a(b^2-c^2)+bc(b-c)=$$

$$(b-c)(a^2-a(b+c)+bc)=(b-c)\big(a(a-b)-c(a-b)\big)=(b-c)(a-b)(a-c)$$

$$So,\qquad LHS=\frac{(a-b)(a-c)(b-c)}{(a-b)(a-c)(b-c)}=1$$

$$\frac{a^2r_a}{(a-b)(a-c)}+\frac{b^2r_b}{(b-a)(b-c)}+\frac{c^2r_c}{(c-b)(c-a)}=$$

$$=F.\frac{s^2}{(s-a)(s-b)(s-c)}=F.\frac{s^2}{F^2}=\frac{s^3}{s.r}=\frac{s^2}{r}\overset{Mitrinovici}{\geq}\frac{27R^2}{4r}$$

4342. In ΔABC the following relationship holds:

$$\sum_{cyc} a(\cos A-\cos B)\leq 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

WLOG: $a\leq b\leq c\Rightarrow \cos A\geq \cos B\geq \cos C$

By rearrangements' inequality:

$$a\cos A+b\cos B+\cos C\leq a\cos B+b\cos C+\cos A$$

$$a(\cos A-\cos B)+b(\cos B-\cos C)+c(\cos C-\cos A)\leq 0$$

$$\sum_{cyc} a\,(\cos A - \cos B) \le 0$$

Equality holds for $a = b = c$.

4343. In ΔABC the following relationship holds:

$$\frac{r_a^3}{r_a^2 + r_a r_b + r_b^2} + \frac{r_b^3}{r_b^2 + r_b r_c + r_c^2} + \frac{r_c^3}{r_c^2 + r_c r_a + r_a^2} \ge \frac{3R}{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Adgozel Adgozalov-Azerbaijan

$$\sum_{cyc} \frac{r_a^3}{r_a^2 + r_a r_b + r_b^2} = \sum_{cyc} \frac{r_a^4}{r_a^3 + r_a^2 r_b + r_a r_b^2}$$

$$\overset{\text{BERGSTROM}}{\ge} \frac{(\sum r_a^2)^2}{\sum_{cyc}(r_a^3 + r_a^2 r_b + r_a r_b^2)} =$$

$$= \frac{(\sum r_a^2)^2}{(\sum r_a)(\sum r_a^2)} =$$

$$= \frac{\sum r_a^2}{\sum r_a} \ge \frac{\frac{27}{4}R^2}{\frac{9}{2}R} = \frac{3R}{2}$$

Equality holds for $a = b = c$.

4344. In ΔABC the following relationship holds:

$$\sum_{cyc} a(w_a - w_b) \le 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

WLOG: $a \le b \le c \Rightarrow w_a \ge w_b \ge w_c$

By rearrangements' inequality:

$$aw_a + bw_b + cw_c \le aw_b + bw_c + cw_a$$

$$a(w_a - w_b) + b(w_b - w_c) + c(w_c - w_a) \le 0$$

$$\sum_{cyc} a(w_a - w_b) \le 0$$

Equality holds for $a=b=c$.

4345. In ΔABC the following relationship holds:

$$\frac{\sin^3\frac{A}{2}}{\sin^2\frac{A}{2}+\sin\frac{A}{2}\sin\frac{B}{2}+\sin^2\frac{B}{2}}+\frac{\sin^3\frac{B}{2}}{\sin^2\frac{B}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+\sin^2\frac{C}{2}}+\frac{\sin^3\frac{C}{2}}{\sin^2\frac{C}{2}+\sin\frac{C}{2}\sin\frac{A}{2}+\sin^2\frac{A}{2}}\geq\frac{1}{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Adgozel Adgozalov-Azerbaijan

$$\sum_{cyc}\frac{\sin^3\frac{A}{2}}{\sin^2\frac{A}{2}+\sin\frac{A}{2}\sin\frac{B}{2}+\sin^2\frac{B}{2}}=\sum_{cyc}\frac{\sin^4\frac{A}{2}}{\sin^3\frac{A}{2}+\sin^2\frac{A}{2}\sin\frac{B}{2}+\sin\frac{A}{2}\sin^2\frac{B}{2}}\geq$$

$$\overset{\text{BERGSTROM}}{\geq}\frac{\left(\sum\sin^2\frac{A}{2}\right)^2}{\sum_{cyc}\left(\sin^3\frac{A}{2}+\sin^2\frac{A}{2}\sin\frac{B}{2}+\sin\frac{A}{2}\sin^2\frac{B}{2}\right)}=$$

$$=\frac{\left(\sum\sin^2\frac{A}{2}\right)^2}{\left(\sum\sin\frac{A}{2}\right)\left(\sum\sin^2\frac{A}{2}\right)}=$$

$$=\frac{\sum\sin^2\frac{A}{2}}{\sum\sin\frac{A}{2}}\geq\frac{\frac{3}{4}}{\frac{3}{2}}=\frac{1}{2}$$

Equality holds for $a=b=c$.

4346. In ΔABC the following relationship holds:

$$\sum_{cyc}a\left(\cos\frac{A}{2}-\cos\frac{B}{2}\right)\leq 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

WLOG: $a\leq b\leq c\Rightarrow\cos\frac{A}{2}\geq\cos\frac{B}{2}\geq\cos\frac{C}{2}$

By rearrangements' inequality:

$$a\cos\frac{A}{2}+b\cos\frac{B}{2}+c\cos\frac{C}{2}\leq a\cos\frac{B}{2}+b\cos\frac{C}{2}+c\cos\frac{A}{2}$$

$$a\left(\cos\frac{A}{2}-\cos\frac{B}{2}\right)+b\left(\cos\frac{B}{2}-\cos\frac{C}{2}\right)+c\left(\cos\frac{C}{2}-\cos\frac{A}{2}\right)\leq 0$$

$$\sum_{cyc}a\left(\cos\frac{A}{2}-\cos\frac{B}{2}\right)\leq 0$$

Equality holds for $a=b=c$.

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4347.

In triangles ABC such that $a \neq b \neq c$ the following relationship holds:

$$\frac{1}{27R^6} < \frac{1}{(a-b)(a-c)a^2bc} + \frac{1}{(b-c)(b-a)b^2ca} + \frac{1}{(c-a)(c-b)c^2ba} < \frac{1}{1728r^6}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{1}{(a-b)(a-c)a^2bc} + \frac{1}{(b-c)(b-a)b^2ca} + \frac{1}{(c-a)(c-b)c^2ba} =$$
$$= \frac{1}{abc}\left(\frac{1}{(a-b)(a-c)a} + \frac{1}{(b-c)(b-a)b} + \frac{1}{(c-a)(c-b)c}\right) =$$
$$= \frac{1}{abc}\left(\frac{-bc(b-c) - ac(c-a) - ab(a-b)}{(a-b)(b-c)(c-a)abc}\right) =$$
$$= \frac{1}{(abc)^2}\left(\frac{-b^2c + c^2b + a^2c - c^2a - a^2b + b^2a}{(a-b)(b-c)(c-a)}\right) =$$
$$= \frac{1}{(abc)^2}\left(\frac{bc(c-b) + a^2(c-b) + a(b^2-c^2)}{(a-b)(b-c)(c-a)}\right) =$$
$$\begin{cases} N = bc(c-b) + a^2(c-b) + a(b^2-c^2) = \\ = (b-c)\big(c(a-b) - a(a-b)\big) = (b-c)(a-b)(c-a) \end{cases}$$
$$= \frac{1}{(abc)^2}\left(\frac{(b-c)(a-b)(c-a)}{(a-b)(b-c)(c-a)}\right) = \frac{1}{(abc)^2}$$
$$\frac{1}{(abc)^2} = \frac{1}{(4RF)^2} = \frac{1}{16R^2F^2} < \frac{1}{16.4r^2.27r^4} = \frac{1}{1728r^6}\ (RHS)$$
$$\frac{1}{(abc)^2} = \frac{1}{16R^2F^2} > \frac{1}{16R^2.\frac{27R^4}{16}} = \frac{1}{27R^6}\ (LHS)$$

4348. In ΔABC the following relationship holds:

$$\sum_{cyc} a\left(\tan\frac{A}{2} - \tan\frac{B}{2}\right) \geq 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

WLOG: $a \leq b \leq c \Rightarrow \tan\frac{A}{2} \leq \tan\frac{B}{2} \leq \tan\frac{C}{2}$

By rearrangements' inequality:

$$a\tan\frac{A}{2} + b\tan\frac{B}{2} + c\tan\frac{C}{2} \geq a\tan\frac{B}{2} + b\tan\frac{C}{2} + c\tan\frac{A}{2}$$

$$a\left(\tan\frac{A}{2}-\tan\frac{B}{2}\right)+b\left(\tan\frac{B}{2}-\tan\frac{C}{2}\right)+c\left(\tan\frac{C}{2}-\tan\frac{A}{2}\right)\geq 0$$

$$\sum_{cyc} a\left(\tan\frac{A}{2}-\tan\frac{B}{2}\right)\geq 0$$

Equality holds for $a=b=c$.

4349. In ΔABC the following relationship holds:

$$\sum_{cyc} h_a\,(w_a-w_b)\geq 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{WLOG: } a\leq b\leq c\Rightarrow\begin{cases}h_a\geq h_b\geq h_c\\ w_a\geq w_b\geq w_c\end{cases}$$

By rearrangements' inequality:

$$h_aw_a+h_bw_b+h_cw_c\geq h_aw_b+h_bw_c+h_cw_a$$

$$h_a(w_a-w_b)+h_b(w_b-w_c)+h_c(w_c-w_a)\geq 0$$

$$\sum_{cyc} h_a\,(w_a-w_b)\geq 0$$

Equality holds for $a=b=c$.

4350. In ΔABC the following relationship holds:

$$\sum_{cyc} a\,(\sin A-\sin B)\geq 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$\sum_{cyc} a\,(\sin A-\sin B)\geq 0\Leftrightarrow\sum_{cyc} a\left(\frac{a}{2R}-\frac{b}{2R}\right)\geq 0$$

$$\Leftrightarrow\sum_{cyc} a(a-b)\geq 0\Leftrightarrow\sum_{cyc} a^2\geq\sum_{cyc} ab$$

$$\Leftrightarrow a^2+b^2+c^2\geq ab+bc+ca\Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}(a-b)^2+\frac{1}{2}(b-c)^2+\frac{1}{2}(c-a)^2\geq 0 \quad \text{(True!)}$$

Equality holds for $a=b=c$.

Solution 2 by Daniel Sitaru – Romania

WLOG: $a\leq b\leq c\Rightarrow \sin A\leq \sin B\leq \sin C$

By rearrangements' inequality:

$$a\sin A+b\sin B+c\sin C\geq a\sin B+b\sin C+c\sin A$$

$$a(\sin A-\sin B)+b(\sin B-\sin C)+c(\sin C-\sin B)\geq 0$$

$$\sum_{cyc} a(\sin A-\sin B)\geq 0$$

Equality holds for $a=b=c$.

4351. In ΔABC the following relationship holds:

$$\sum_{cyc} a\left(\cot\frac{A}{2}-\cot\frac{B}{2}\right)\leq 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

WLOG: $a\leq b\leq c\Rightarrow \cot\frac{A}{2}\geq \cot\frac{B}{2}\geq \cot\frac{C}{2}$

By rearrangements' inequality:

$$a\cot\frac{A}{2}+b\cot\frac{B}{2}+c\cot\frac{C}{2}\leq a\cot\frac{B}{2}+b\cot\frac{C}{2}+\cot\frac{A}{2}$$

$$a\left(a\cot\frac{A}{2}-\cot\frac{B}{2}\right)+b\left(\cot\frac{B}{2}-\cot\frac{C}{2}\right)+c\left(\cot\frac{C}{2}-\cot\frac{A}{2}\right)\leq 0$$

$$\sum_{cyc} a\left(\cot\frac{A}{2}-\cot\frac{B}{2}\right)\leq 0$$

Equality holds for $a=b=c$.

4352. In ΔABC the following relationship holds:

$$\sum_{cyc} a(r_a-r_b)\leq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc} a(m_a - m_b) \le 0, \qquad ar_a + br_b + cr_c \ge ar_b + br_c + cr_a \ (*)$$

$$Let\ us\ prove\ that:$$
$$Wlog\ :\ a \le b \le c \to r_a \le r_b \le r_c$$
$$These\ conditions\ are\ suitable\ for\ Rearrangement\ inequality:$$
$$A_{max} = ar_a + br_b + cr_c, \qquad C_{mixed} = ar_b + br_c + cr_a$$
$$So:\ A_{max} \ge C_{mixed}\ (Proved)$$
$$Equality\ holds\ iff\ a = b = c$$

4353. In ΔABC the following relationship holds:

$$\sum_{cyc} \sin\left(\frac{A}{2}\right)\cos(A) \le \frac{3}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc} \sin\left(\frac{A}{2}\right)\cos(A) = \sum_{cyc} \sin\left(\frac{A}{2}\right)\left(1 - 2sin^2\left(\frac{A}{2}\right)\right) = \sum_{cyc}\left(\sin\left(\frac{A}{2}\right) - 2sin^3\left(\frac{A}{2}\right)\right)$$

$$Let\ \sin\left(\frac{A}{2}\right) = x\ , \qquad x\epsilon(0;1), \qquad f(x) = x - 2x^3 \to \quad f''(x) = -12x < 0$$

$$So\ f(x)\ is\ concave\ function:\ By\ Jensen's\ Inequality$$

$$\sum_{cyc}\left(\sin\left(\frac{A}{2}\right) - 2sin^3\left(\frac{A}{2}\right)\right) \le 3\left(sin\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3} - 2sin^3\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right) =$$

$$= 3\left(\frac{1}{2} - 2\left(\frac{1}{2}\right)^3\right) = \frac{3}{4}$$

$$Equality\ holds\ iff\ a = b = c$$

4354. In acute ΔABC the following relationship holds:

$$\sum_{cyc} \tan\left(\frac{A}{2}\right)\sec(A) \ge 2\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc} \tan\left(\frac{A}{2}\right)\sec(A) = \sum_{cyc} \tan\left(\frac{A}{2}\right).\frac{1 + tan^2\left(\frac{A}{2}\right)}{1 - tan^2\left(\frac{A}{2}\right)} = \sum_{cyc}\frac{\tan\left(\frac{A}{2}\right) + tan^3\left(\frac{A}{2}\right)}{1 - tan^2\left(\frac{A}{2}\right)}$$

$$Let\ \tan\left(\frac{A}{2}\right)=x\ ,\qquad x\epsilon(0;1),\qquad f(x)=\frac{x+x^3}{1-x^2}\rightarrow\quad f''(x)=\frac{4x^3+12x}{(1-x^2)^3}>0$$

$$So\ f(x)\ is\ convex\ function:\ By\ Jensen's\ Inequality$$

$$\sum_{cyc}\frac{\tan\left(\frac{A}{2}\right)+tan^3\left(\frac{A}{2}\right)}{1-tan^2\left(\frac{A}{2}\right)}\geq 3\left(\frac{tan\frac{A+B+C}{6}+tan^3\frac{A+B+C}{6}}{1-tan^2\frac{A+B+C}{6}}\right)=3\left(\frac{\frac{\sqrt{3}}{3}+\left(\frac{\sqrt{3}}{3}\right)^3}{1-\left(\frac{\sqrt{3}}{3}\right)^2}\right)=$$

$$=3.\frac{36\sqrt{3}}{54}=2\sqrt{3}$$

$$Equality\ holds\ if\ a=b=c$$

4355. In ΔABC the following relationship holds:

$$\sum_{cyc} m_a\left(\cos\frac{A}{2}-\cos\frac{B}{2}\right)\geq 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{WLOG: } a\leq b\leq c\Rightarrow\begin{cases} m_a\geq m_b\geq m_c\\ \cos\frac{A}{2}\geq\cos\frac{B}{2}\geq\cos\frac{C}{2}\end{cases}$$

By rearrangements' inequality:

$$m_a\cos\frac{A}{2}+m_b\cos\frac{B}{2}+m_c\cos\frac{C}{2}\geq m_a\cos\frac{B}{2}+m_b\cos\frac{C}{2}+m_c\cos\frac{A}{2}$$

$$m_a\left(\cos\frac{A}{2}-\cos\frac{B}{2}\right)+m_b\left(\cos\frac{B}{2}-\cos\frac{C}{2}\right)+m_c\left(\cos\frac{C}{2}-\cos\frac{A}{2}\right)\geq 0$$

Equality holds for $a=b=c$.

4356. In ΔABC the following relationship holds:

$$\sum_{cyc} w_a\left(\tan\frac{A}{2}-\tan\frac{B}{2}\right)\leq 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{WLOG: } a\leq b\leq c\Rightarrow\begin{cases} w_a\geq w_b\geq w_c\\ \tan\frac{A}{2}\leq\tan\frac{B}{2}\leq\tan\frac{C}{2}\end{cases}$$

By rearrangements' inequality:

$$w_a\tan\frac{A}{2}+w_b\tan\frac{B}{2}+w_c\tan\frac{C}{2}\le w_a\tan\frac{B}{2}+w_b\tan\frac{C}{2}+w_c\tan\frac{A}{2}$$

$$w_a\left(\tan\frac{A}{2}-\tan\frac{B}{2}\right)+w_b\left(\tan\frac{B}{2}-\tan\frac{C}{2}\right)+w_c\left(\tan\frac{C}{2}-\tan\frac{A}{2}\right)\le 0$$

$$\sum_{cyc} w_a\left(\tan\frac{A}{2}-\tan\frac{B}{2}\right)\le 0$$

Equality holds for $a=b=c$.

4357. In ΔABC the following relationship holds:

$$\sum_{cyc} h_a(r_a-r_b)\le 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

WLOG: $a\le b\le c\Rightarrow \frac{1}{a}\ge\frac{1}{b}\ge\frac{1}{c}\Rightarrow\frac{2F}{a}\ge\frac{2F}{b}\ge\frac{2F}{c}\Rightarrow h_a\ge h_b\ge h_c$

$$a\le b\le c\Rightarrow -a\ge -b\ge -c\Rightarrow s-a\ge s-b\ge s-c$$

$$\Rightarrow\frac{1}{s-a}\le\frac{1}{s-b}\le\frac{1}{s-c}\Rightarrow\frac{F}{s-a}\le\frac{F}{s-b}\le\frac{F}{s-c}\Rightarrow r_a\le r_b\le r_c$$

By rearrangements' inequality:

$$h_ar_a+h_br_b+h_cr_c\le h_ar_b+h_br_c+h_cr_a$$

$$h_a(r_a-r_b)+h_b(r_b-r_c)+h_c(r_c-r_a)\le 0$$

$$\sum_{cyc} h_a(r_a-r_b)\le 0$$

Equality holds for $a=b=c$.

4358. In ΔABC the following relationship holds:

$$\sum_{cyc} w_a\left(\sin\frac{A}{2}-\sin\frac{B}{2}\right)\le 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

WLOG: $a\le b\le c\Rightarrow\begin{cases} w_a\ge w_b\ge w_c\\ \sin\frac{A}{2}\le\sin\frac{B}{2}\le\sin\frac{C}{2}\end{cases}$

By rearrangements' inequality:

$$w_a\sin\frac{A}{2}+w_b\sin\frac{B}{2}+w_c\sin\frac{C}{2}\le w_a\sin\frac{B}{2}+w_b\sin\frac{C}{2}+w_c\sin\frac{A}{2}$$

$$w_a\left(\sin\frac{A}{2}-\sin\frac{B}{2}\right)+w_b\left(\sin\frac{B}{2}-\sin\frac{C}{2}\right)+w_c\left(\sin\frac{C}{2}-\sin\frac{A}{2}\right)\le 0$$

$$\sum_{cyc} w_a\left(\sin\frac{A}{2}-\sin\frac{B}{2}\right)\le 0$$

Equality holds for $a=b=c$.

4359. In ΔABC the following relationship holds:

$$h_a+h_b+h_c=9r+r\sum_{cyc}\left(\sqrt{\frac{b}{c}}-\sqrt{\frac{c}{b}}\right)^2$$

Proposed by Bogdan Fuștei-Romania

Solution by Daniel Sitaru-Romania

$$h_a+h_b+h_c=9r+r\sum_{cyc}\left(\sqrt{\frac{b}{c}}-\sqrt{\frac{c}{b}}\right)^2$$

$$\frac{2F}{a}+\frac{2F}{b}+\frac{2F}{c}=\frac{9F}{s}+\frac{F}{s}\sum_{cyc}\left(\frac{b}{c}+\frac{c}{b}-2\right)$$

$$2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{3}{s}+\frac{1}{s}\sum_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right)$$

$$2s\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=3+\sum_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right)$$

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=3+\sum_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right)$$

$$3+\sum_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right)=3+\sum_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right)$$

4360. In ΔABC the following relationship holds:

$$\sum_{cyc} r_a\left(\cos\frac{A}{2}-\cos\frac{B}{2}\right)\le 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

WLOG: $a\le b\le c\Rightarrow -a\ge -b\ge -c\Rightarrow$

$$\Rightarrow s-a\geq s-b\geq s-c\Rightarrow \frac{1}{s-a}\leq\frac{1}{s-b}\leq\frac{1}{s-c}\Rightarrow \frac{F}{s-a}\leq\frac{F}{s-b}\leq\frac{F}{s-c}$$

$$\Rightarrow r_a\leq r_b\leq r_c,\qquad a\leq b\leq c\Rightarrow \cos\frac{A}{2}\geq\cos\frac{B}{2}\geq\cos\frac{C}{2}$$

By rearrangements' inequality:

$$r_a\cos\frac{A}{2}+r_b\cos\frac{B}{2}+r_c\cos\frac{C}{2}\leq r_a\cos\frac{B}{2}+r_b\cos\frac{C}{2}+r_c\cos\frac{A}{2}$$

$$r_a\left(\cos\frac{A}{2}-\cos\frac{B}{2}\right)+r_b\left(\cos\frac{B}{2}-\cos\frac{C}{2}\right)+r_c\left(\cos\frac{C}{2}-\cos\frac{A}{2}\right)\leq 0$$

$$\sum_{cyc} r_a\left(\cos\frac{A}{2}-\cos\frac{B}{2}\right)\leq 0$$

Equality holds for $a=b=c$.

4361. In any ΔABC the following relationship holds :

$$w_a+w_b+w_c\leq\frac{h_a r_a}{w_a}+\frac{h_b r_b}{w_b}+\frac{h_c r_c}{w_c}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}\frac{h_a r_a}{w_a}\right)^2=\sum_{cyc}\left(\cos^2\frac{B-C}{2}\cdot\frac{r^2s^2}{(s-a)^2}\right)+2\sum_{cyc}\left(\cos\frac{B-C}{2}\cos\frac{C-A}{2}\cdot r_a r_b\right)$$

$$=\sum_{cyc}\left(\frac{\left(s^2+(s-a)^2+2s(s-a)\right)\sin^2\frac{A}{2}}{16R^2\cdot\sin^2\frac{A}{2}\cdot\frac{s(s-a)}{bc}}\cdot\frac{r^2s^2}{(s-a)^2}\right)+2\cos\frac{B-C}{2}\cos\frac{C-A}{2}\cos\frac{A-B}{2}\cdot\sum_{cyc}\frac{r_a r_b}{\cos\frac{A-B}{2}}$$

$$\geq r^2s^2\cdot\frac{s}{16R^2}\cdot\sum_{cyc}\frac{bc}{(s-a)^3}+r^2s^2\cdot\frac{1}{16R^2s}\cdot\sum_{cyc}\frac{bc}{s-a}+\frac{r^2s^2}{8R^2}\cdot\sum_{cyc}\frac{bc}{(s-a)^2}+2\cdot\frac{2s(s^2+2Rr+r^2)}{4Rrs}\cdot\frac{r}{4R}\cdot\sum_{cyc}r_b r_c\quad\left(\because 0<\cos\frac{A-B}{2}\leq 1\ \text{and analogs}\right)\rightarrow(1)$$

$$\text{Now, } \sum_{cyc}\frac{bc}{(s-a)^3}=\frac{1}{r^6s^3}\cdot\sum_{cyc}(bc(-s^2+sa+bc)^3)=$$

$$\frac{1}{r^6s^3}\cdot\sum_{cyc}\left(bc\left(\begin{matrix}a^3s^3+3a^2bcs^2-3a^2s^4+3ab^2c^2s-6abcs^3+3as^5+b^3c^3-\\3b^2c^2s^2+3bcs^4-s^6\end{matrix}\right)\right)$$

$$8Rrs^4(s^2-4Rr-r^2)+144R^2r^2s^4-24Rrs^6+12Rrs^2((s^2+4Rr+r^2)^2-16Rrs^2)$$
$$-24Rrs^4(s^2+4Rr+r^2)+36Rrs^6+$$
$$((s^2+4Rr+r^2)^2-16Rrs^2)^2-64R^2r^2s^2(s^2-4Rr-r^2)-$$
$$3s^2\left((s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\right)+3s^4((s^2+4Rr+r^2)^2-16Rrs^2)$$
$$=\frac{-s^6(s^2+4Rr+r^2)}{r^6s^3}$$

$$\rightarrow (2);\ again,\ \sum_{cyc}\frac{bc}{(s-a)^2}=\frac{1}{r^4s^2}\cdot\sum_{cyc}(bc(-s^2+sa+bc)^2)$$

$$=\frac{1}{r^4s^2}\cdot\sum_{cyc}\left(bc(a^2s^2+2abcs-2as^3+b^2c^2-2bcs^2+s^4)\right)$$

$$=\frac{1}{r^4s^2}\cdot\left(\begin{matrix}8Rrs^4+8Rrs^2(s^2+4Rr+r^2)-24Rrs^4+(s^2+4Rr+r^2)^3-\\24Rrs^2(s^2+2Rr+r^2)-2s^2((s^2+4Rr+r^2)^2-16Rrs^2)+s^4(s^2+4Rr+r^2)\end{matrix}\right)$$

$$\rightarrow (3);\ also,\ \sum\frac{bc}{s-a}=\frac{1}{r^2s}\cdot\sum_{cyc}\left(bc(-s^2+sa+bc)\right)$$

$$=\frac{-s^2(s^2+4Rr+r^2)+12Rrs^2+(s^2+4Rr+r^2)^2-16Rrs^2}{r^2s}\rightarrow (4)\ and\ so,$$

$$via\ (1)\ and\ via\ (2)+(3)+(4),\ we\ get,\ via\ simplification:\left(\sum_{cyc}\frac{h_ar_a}{w_a}\right)^2\geq$$

$$\frac{s^4-(12R^2+4Rr-2r^2)s^2+64R^4+96R^3r+52R^2r^2+12Rr^3+r^4}{4R^2}\overset{?}{\geq}$$

$$s^2+21Rr+12r^2\Leftrightarrow$$

$$s^4-(16R^2+4Rr-2r^2)s^2+64R^4+12R^3r+4R^2r^2+12Rr^3+r^4\ \underset{(*)}{\overset{?}{\geq}}\ 0$$

$$and\ \because (s^2-16Rr+5r^2)^2\overset{Gerretsen}{\geq}0\ \therefore\ in\ order\ to\ prove\ (*),\ it\ suffices\ to\ prove:$$

$$LHS\ of\ (*)\overset{?}{\geq}(s^2-16Rr+5r^2)^2\Leftrightarrow$$

$$16R^4+3R^3r-63R^2r^2+43Rr^3-6r^4\ \underset{(**)}{\overset{?}{\geq}}\ (4R^2-7Rr+2r^2)s^2$$

$$Finally,\ RHS\ of\ (**)\overset{Gerretsen}{\leq}(4R^2-7Rr+2r^2)(4R^2+4Rr+3r^2)\overset{?}{\leq}LHS\ of\ (**)$$
$$\Leftrightarrow 15t^3-55t^2+56t-12\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(15t^2-25t+6)\overset{?}{\geq}0\rightarrow true$$

$$\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \left(\sum_{cyc}\frac{h_a r_a}{w_a}\right)^2 \geq s^2 + 21Rr + 12r^2 \geq \left(\sum_{cyc} w_a\right)^2$$

(Reference : Inequality in Triangle by Dang Ngoc Minh − 90; published at www. ssmrmh. ro) and so,

$$w_a + w_b + w_c \leq \frac{h_a r_a}{w_a} + \frac{h_b r_b}{w_b} + \frac{h_c r_c}{w_c} \; \forall \Delta ABC, '' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4362. In ΔABC the following relationship holds:

$$\sum_{cyc} w_a(\sin A - \sin B) \leq 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{WLOG: } a \leq b \leq c \Rightarrow \begin{cases} w_a \geq w_b \geq w_c \\ \sin A \leq \sin B \leq \sin C \end{cases}$$

By rearrangements' inequality:

$$w_a \sin A + w_b \sin B + w_c \sin C \leq w_a \sin B + w_b \sin C + w_c \sin A$$

$$w_a(\sin A - \sin B) + w_b(\sin B - \sin C) + w_c(\sin C - \sin A) \leq 0$$

$$\sum_{cyc} w_a(\sin A - \sin B) \leq 0$$

Equality holds for $a = b = c$.

4363. In Δ ABC holds :

$$\frac{\sqrt{3}}{6} \cdot \sum_{cyc}(a m_b m_c) \geq \frac{1}{8} \cdot \prod_{cyc}(m_a + m_b) \geq \frac{1}{4}\sum_{cyc}(a^2 m_a) \geq m_a m_b m_c$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}\left(\frac{1}{3.\sqrt{3}} m_a bc\right)\right)^2 =$$

$$\frac{1}{27}\left(\frac{1}{4}\sum_{cyc}\left(b^2c^2\left(2\sum_{cyc}a^2-3a^2\right)\right)+4Rrs\sum_{cyc}(a.2m_bm_c)\right)\geq$$

$$\frac{1}{27}\left((s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)+4Rrs\sum_{cyc}\left(a\left(\frac{9}{4}a^2+h_a^2-\frac{(3a^2+\sum_{cyc}a^2)}{4}\right)\right)\right)$$

(Reference : Solution to Inequality in Triangle by Dang Ngoc Minh − 393; published at www. ssmrmh. ro)

$$=\frac{1}{27}\left((s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)+4Rrs^2.\frac{3R(s^2-6Rr-3r^2)-R(s^2-4Rr-r^2)+r(s^2+4Rr+r^2)}{R}\right)$$

$$\overset{?}{\geq}\left(\frac{1}{16}\prod_{cyc}(b+c)\right)^2=\frac{s^2(s^2+2Rr+r^2)^2}{64}$$

$\Leftrightarrow 37s^6-(364Rr-266r^2)s^4-r^2(2924R^2+620Rr-165r^2)s^2-r^3(4R+r)^3\overset{?}{\geq}0$ and $\because (s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\geq}0\ \therefore$ to prove ①, it suffices to prove :

$$\text{LHS of ①}\overset{?}{\geq}37(s^2-16Rr+5r^2)^3\Leftrightarrow(1412R-289r)s^4-10r(3134R^2-1714Rr+261r^2)s^2+r^2(16(9216R^3-9072R^2r+2727Rr^2-293r^3)-r^3)\overset{?}{\geq}0$$ and

$\because (1412R-289r)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0\ \therefore$ in order to prove ②, it suffices to prove : LHS of $②\overset{?}{\geq}(1412R-289r)(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow(3461R^2-1557Rr+70r^2)s^2\overset{?}{\geq}r(16(3344R^3-2418R^2r+592Rr^2-40r^3)+5Rr^2+6r^3)$$ and finally,

$(3461R^2-1557Rr+70r^2)s^2\overset{\text{Gerretsen}}{\geq}(3461R^2-1557Rr+70r^2)(16Rr-5r^2)$

$\overset{?}{\geq}$ RHS of ③ $\Leftrightarrow 1872t^3-3529t^2-572t+284\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$

$\Leftrightarrow(t-2)(1872t^2+215t-142)\overset{?}{\geq}0\rightarrow$ true $\because t\overset{\text{Euler}}{\geq}2\Rightarrow③\Rightarrow②\Rightarrow①$ is true

$\therefore\sum_{cyc}\left(\frac{1}{3.\sqrt{3}}m_abc\right)\geq\frac{1}{16}\prod_{cyc}(b+c)$ & implementing it on a ΔXYZ with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$, whose medians as a consequence of trivial calculations are $\frac{a}{2},\frac{b}{2},\frac{c}{2}$,

we arrive at : $\frac{4}{54.\sqrt{3}}.\sum_{cyc}(am_bm_c)\geq\frac{1}{16}.\frac{8}{27}.\prod_{cyc}(m_a+m_b)$

$\therefore\boxed{\frac{\sqrt{3}}{6}.\sum_{cyc}(am_bm_c)\geq\frac{1}{8}.\prod_{cyc}(m_a+m_b)}$; again, $\frac{1}{32}\prod_{cyc}(b+c)\overset{?}{\geq}\frac{1}{9}\sum_{cyc}(am_a^2)$

$$=\frac{1}{36}\sum_{cyc}\left(a\left(2\sum_{cyc}a^2-3a^2\right)\right)\Leftrightarrow\frac{2s(s^2+2Rr+r^2)}{32}\overset{?}{\geq}$$

$$\frac{1}{36}\left(4(s^2-4Rr-r^2)(2s)-(2s)3(s^2-6Rr-3r^2)\right)$$

$\Leftrightarrow(s^2-16Rr+5r^2)+18r(R-2r)\overset{?}{\geq}0\rightarrow$ true via Gerretsen and Euler

$\therefore\frac{1}{32}\prod_{cyc}(b+c)\geq\frac{1}{9}\sum_{cyc}(am_a^2)$ and implementing it on triangle XYZ,

we get : $\boxed{\frac{1}{8}.\prod_{cyc}(m_a+m_b)\geq\frac{1}{4}\sum_{cyc}(a^2m_a)}$ and finally, $\frac{1}{9}\sum_{cyc}(am_a^2)\overset{?}{\geq}\frac{abc}{4}$

$$\Leftrightarrow\frac{1}{36}\left(4(s^2-4Rr-r^2)(2s)-(2s)3(s^2-6Rr-3r^2)\right)\overset{?}{\geq}Rrs\Leftrightarrow$$

$s^2-16Rr+5r^2\overset{?}{\geq}0\rightarrow$ true via Gerretsen $\therefore\frac{1}{9}\sum_{cyc}(am_a^2)\geq\frac{abc}{4}$ & implementing

it on triangle XYZ, we get : $\boxed{\frac{1}{4}\sum_{cyc}(a^2m_a)\geq m_am_bm_c}$ and so,

$$\frac{\sqrt{3}}{6}.\sum_{cyc}(am_bm_c)\geq\frac{1}{8}.\prod_{cyc}(m_a+m_b)\geq\frac{1}{4}\sum_{cyc}(a^2m_a)\geq m_am_bm_c\ \forall\ \Delta ABC,$$

$''=''$ iff ΔABC is equilateral (QED)

4364. In ΔABC the following relationship holds:

$$\sum_{cyc}r_a\left(\sin\frac{A}{2}-\sin\frac{B}{2}\right)\geq 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

WLOG: $a\leq b\leq c\Rightarrow -a\geq -b\geq -c\Rightarrow s-a\geq s-b\geq s-c$

$$\frac{1}{s-a}\leq\frac{1}{s-b}\leq\frac{1}{s-c}\Rightarrow\frac{F}{s-a}\leq\frac{F}{s-b}\leq\frac{F}{s-c}\Rightarrow r_a\leq r_b\leq r_c$$

$$a \le b \le c \Rightarrow \sin\frac{A}{2} \le \sin\frac{B}{2} \le \sin\frac{C}{2}$$

By rearrangements' inequality:

$$r_a \sin\frac{A}{2} + r_b \sin\frac{B}{2} + r_c \sin\frac{C}{2} \le r_a \sin\frac{B}{2} + r_b \sin\frac{C}{2} + r_c \sin\frac{A}{2}$$

$$r_a\left(\sin\frac{A}{2} - \sin\frac{B}{2}\right) + r_b\left(\sin\frac{B}{2} - \sin\frac{C}{2}\right) + r_c\left(\sin\frac{C}{2} - \sin\frac{A}{2}\right) \le 0$$

$$\sum_{cyc} r_a\left(\sin\frac{A}{2} - \sin\frac{B}{2}\right) \le 0$$

Equality holds for $a = b = c$.

4365. In any ΔABC the following relationship holds :

$$w_a + w_b + w_c \le \frac{h_a r_a}{w_a} + \frac{h_b r_b}{w_b} + \frac{h_c r_c}{w_c}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}\frac{h_a r_a}{w_a}\right)^2 = \sum_{cyc}\left(\cos^2\frac{B-C}{2}\cdot\frac{r^2s^2}{(s-a)^2}\right) + 2\sum_{cyc}\left(\cos\frac{B-C}{2}\cos\frac{C-A}{2}\cdot r_a r_b\right)$$

$$= \sum_{cyc}\left(\frac{\left(s^2+(s-a)^2+2s(s-a)\right)\sin^2\frac{A}{2}}{16R^2\cdot\sin^2\frac{A}{2}\cdot\frac{s(s-a)}{bc}}\cdot\frac{r^2s^2}{(s-a)^2}\right) + 2\cos\frac{B-C}{2}\cos\frac{C-A}{2}\cos\frac{A-B}{2}\cdot\sum_{cyc}\frac{r_a r_b}{\cos\frac{A-B}{2}}$$

$$\ge r^2s^2\cdot\frac{s}{16R^2}\cdot\sum_{cyc}\frac{bc}{(s-a)^3} + r^2s^2\cdot\frac{1}{16R^2s}\cdot\sum_{cyc}\frac{bc}{s-a} + \frac{r^2s^2}{8R^2}\cdot\sum_{cyc}\frac{bc}{(s-a)^2} + 2\cdot\frac{2s(s^2+2Rr+r^2)}{4Rrs}\cdot\frac{r}{4R}\cdot\sum_{cyc} r_b r_c \quad \left(\because 0 < \cos\frac{A-B}{2} \le 1 \text{ and analogs}\right) \rightarrow (1)$$

$$\text{Now, } \sum_{cyc}\frac{bc}{(s-a)^3} = \frac{1}{r^6s^3}\cdot\sum_{cyc}(bc(-s^2+sa+bc)^3) =$$

$$\frac{1}{r^6s^3}\cdot\sum_{cyc}\left(bc\left(\begin{matrix}a^3s^3+3a^2bcs^2-3a^2s^4+3ab^2c^2s-6abcs^3+3as^5+b^3c^3-\\3b^2c^2s^2+3bcs^4-s^6\end{matrix}\right)\right)$$

$$8Rrs^4(s^2-4Rr-r^2)+144R^2r^2s^4-24Rrs^6+12Rrs^2((s^2+4Rr+r^2)^2-16Rrs^2)$$
$$-24Rrs^4(s^2+4Rr+r^2)+36Rrs^6+$$
$$((s^2+4Rr+r^2)^2-16Rrs^2)^2-64R^2r^2s^2(s^2-4Rr-r^2)-$$
$$3s^2\left((s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\right)+3s^4((s^2+4Rr+r^2)^2-16Rrs^2)$$
$$=\frac{-s^6(s^2+4Rr+r^2)}{r^6s^3}$$

$\rightarrow$ ②; *again*, $\sum_{cyc}\frac{bc}{(s-a)^2}=\frac{1}{r^4s^2}\cdot\sum_{cyc}(bc(-s^2+sa+bc)^2)$

$$=\frac{1}{r^4s^2}\cdot\sum_{cyc}\left(bc(a^2s^2+2abcs-2as^3+b^2c^2-2bcs^2+s^4)\right)$$

$$=\frac{1}{r^4s^2}\cdot\left(\begin{matrix}8Rrs^4+8Rrs^2(s^2+4Rr+r^2)-24Rrs^4+(s^2+4Rr+r^2)^3-\\24Rrs^2(s^2+2Rr+r^2)-2s^2((s^2+4Rr+r^2)^2-16Rrs^2)+s^4(s^2+4Rr+r^2)\end{matrix}\right)$$

$\rightarrow$ ③; *also*, $\sum\frac{bc}{s-a}=\frac{1}{r^2s}\cdot\sum_{cyc}\left(bc(-s^2+sa+bc)\right)$

$$=\frac{-s^2(s^2+4Rr+r^2)+12Rrs^2+(s^2+4Rr+r^2)^2-16Rrs^2}{r^2s}\rightarrow④\ \textit{and so,}$$

via ① *and via* ② + ③ + ④, *we get, via simplification* : $\left(\sum_{cyc}\frac{h_ar_a}{w_a}\right)^2\geq$

$$\frac{s^4-(12R^2+4Rr-2r^2)s^2+64R^4+96R^3r+52R^2r^2+12Rr^3+r^4}{4R^2}\overset{?}{\geq}$$
$$s^2+21Rr+12r^2\Leftrightarrow$$
$$s^4-(16R^2+4Rr-2r^2)s^2+64R^4+12R^3r+4R^2r^2+12Rr^3+r^4\overset{?}{\underset{(*)}{\geq}}0$$

and $\because (s^2-16Rr+5r^2)^2\overset{Gerretsen}{\geq}0\ \therefore$ *in order to prove* $(*)$, *it suffices to prove* :

$$\text{LHS of }(*)\overset{?}{\geq}(s^2-16Rr+5r^2)^2\Leftrightarrow$$
$$16R^4+3R^3r-63R^2r^2+43Rr^3-6r^4\overset{?}{\underset{(**)}{\geq}}(4R^2-7Rr+2r^2)s^2$$

Finally, RHS of $(**)\overset{Gerretsen}{\leq}(4R^2-7Rr+2r^2)(4R^2+4Rr+3r^2)\overset{?}{\leq}$ LHS of $(**)$

$$\Leftrightarrow 15t^3-55t^2+56t-12\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(15t^2-25t+6)\overset{?}{\geq}0\rightarrow\text{true}$$

$$\because t\overset{Euler}{\geq}2\Rightarrow(**)\Rightarrow(*)\text{ is true}\ \therefore\left(\sum_{cyc}\frac{h_ar_a}{w_a}\right)^2\geq s^2+21Rr+12r^2\geq\left(\sum_{cyc}w_a\right)^2$$

(Reference : Inequality in Triangle by Dang Ngoc Minh − 90; published at www. ssmrmh. ro) and so,

$$w_a + w_b + w_c \le \frac{h_a r_a}{w_a} + \frac{h_b r_b}{w_b} + \frac{h_c r_c}{w_c} \ \forall \Delta ABC, '' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4366. In acute ΔABC the following relationship holds:

$$\sum_{cyc} w_a(\cot A - \cot B) \ge 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{WLOG: } a \le b \le c \Rightarrow \begin{cases} w_a \ge w_b \ge w_c \\ \cot A \ge \cot B \ge \cot C \end{cases}$$

By rearrangements' inequality:

$$w_a \cot A + w_b \cot B + w_c \cot C \ge w_a \cot B + w_b \cot C + w_c \cot A$$

$$w_a(\cot A - \cot B) + w_b(\cot B - \cot C) + w_c(\cot C - \cot A) \ge 0$$

$$\sum_{cyc} w_a (\cot A - \cot B)$$

Equality holds for $a = b = c$.

4367. In any Δ ABC the following relationship holds :

$$\sum_{cyc} \frac{h_b h_c}{h_a} \le \sum_{cyc} m_a \le \sum_{cyc} \frac{w_a h_a}{r_a}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$2F \sum_{cyc} \frac{m_a}{m_b m_c} \overset{?}{\le} \sum_{cyc} a \Leftrightarrow 4F^2 \left(\sum_{cyc} m_a^2\right)^2 \overset{?}{\le} 4s^2 . (m_a m_b m_c)^2 \Leftrightarrow$$

$$\frac{s^6 - s^4(12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3}{16} \overset{?}{\ge}$$

$$r^2 . \frac{9}{4}(s^2 - 4Rr - r^2)^2$$

(Reference : Solution to Inequality in Triangle by Dang Ngoc Minh − 124; published at www. ssmrmh. ro) $\Leftrightarrow$

$$s^6-(12Rr+3r^2)s^4-r^2(60R^2-168Rr-39r^2)s^2-$$
$$r^3(64R^3+624R^2r+300Rr^2+37r^3)\overset{?}{\geq}_{①}0$$

and $\because (s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove ①, it suffices to prove :

$$\text{LHS of ①} \overset{?}{\geq} (s^2-16Rr+5r^2)^3 \Leftrightarrow (2R-r)s^4-r(46R^2-36Rr+2r^2)s^2+$$
$$r^2(224R^3-248R^2r+50Rr^2-9r^3)\overset{?}{\geq}_{②}0 \text{ and } \because (s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0$$

$\therefore$ to prove ②, it suffices to prove : $\text{LHS of ②} \overset{?}{\geq} (2R-r)(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow (9R^2-8Rr+4r^2)s^2\overset{?}{\geq}_{③} r(144R^3-164R^2r+80Rr^2-8r^3) \text{ and finally,}$$

via Rouche + Euler, $\text{LHS of ③} \geq (9R^2-8Rr+4r^2)\left(16Rr-5r^2+\frac{r^2(R-2r)}{R-r}\right)$

$\overset{?}{\geq} \text{RHS of ③} \Leftrightarrow r^3(R-2r)(7R-2r) \overset{?}{\geq} 0 \rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow ③ \Rightarrow ② \Rightarrow ①$ is true $\therefore 2F\sum_{cyc}\frac{m_a}{m_bm_c} \leq \sum_{cyc} a$ and implementing it on a triangle XYZ with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$, whose medians and area as a consequence of trivial calculations are $\frac{a}{2},\frac{b}{2},\frac{c}{2}$ and $\frac{F}{3}$ respectively, we arrive at : $\frac{2F}{3}.\sum_{cyc}\frac{\left(\frac{a}{2}\right)}{\left(\frac{bc}{4}\right)} \leq \frac{2}{3}\sum_{cyc} m_a \Rightarrow 2F\frac{\frac{2F}{h_a}}{\frac{4F^2}{h_bh_c}} \leq$

$$\sum_{cyc} m_a \Rightarrow \boxed{\sum_{cyc}\frac{h_bh_c}{h_a} \leq \sum_{cyc} m_a} \text{ and now, } \sum_{cyc}\frac{w_ah_a}{r_a}=\sum_{cyc}\left(\frac{w_a}{2Rr}.AI^2\right)$$

$$=\frac{1}{2Rr}\sum_{cyc}\left(w_a(bc-4Rr)\right) \overset{\text{Chebyshev}}{\geq} \frac{1}{6Rr}\left(\sum_{cyc} w_a\right).\sum_{cyc}(bc-4Rr)$$

($\because$ WLOG assuming $a \geq b \geq c \Rightarrow w_a \leq w_b \leq w_c$ and $bc \leq ca \leq ab$)

$$\overset{\text{Dang Ngoc Minh}}{\geq} \frac{\sqrt{s^2+12Rr+30r^2}}{6Rr}.(s^2-8Rr+r^2)$$

$$\left(\begin{matrix}\text{Reference : Inequality in Triangle by Dang Ngoc Minh} - 90;\\ \text{published at www. ssmrmh. ro}\end{matrix}\right) \overset{?}{\geq} \sqrt{4s^2-16Rr+5r^2}$$

$$\Leftrightarrow s^6-(4Rr-32r^2)s^4-r^2(272R^2+472Rr-61r^2)s^2+$$
$$r^3(1344R^3+1548R^2r-468Rr^2+30r^3)\overset{?}{\geq}_{(*)}0 \ \& \ \because (s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0$$

$\therefore$ in order to prove $(*)$, it suffices to prove : $\text{LHS of } (*) \overset{?}{\geq} (s^2-16Rr+5r^2)^3$

$$\Leftrightarrow (44R+17r)s^4-r(1040R^2-8Rr+14r^2)s^2+$$

$$r^2(5440R^3-22928R^2r+732Rr^2-95r^3)\overset{?}{\geq}0;\because (s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0$$

$$\therefore \text{in order to prove } (**), \text{it suffices to prove : LHS of } (**)\overset{?}{\geq}(44R+17r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (92R^2+28Rr-46r^2)s^2\overset{?}{\geq}r(1456R^3-99R^2r-588Rr^2+130r^3)\ \textit{and}$$

$$\textit{finally},(92R^2+28Rr-46r^2)s^2\overset{\text{Gerretsen}}{\geq}(92R^2+28Rr-46r^2)(16Rr-5r^2)$$

$$\overset{?}{\geq}\text{RHS of } (***)\Leftrightarrow 16t^3+87t^2-288t+100\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(16t^2+119t-50)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow (***)\Rightarrow (**)\Rightarrow (*)\text{ is true}$$

$$\therefore \sum_{cyc}\frac{w_ah_a}{r_a}\geq\sqrt{4s^2-16Rr+5r^2}\overset{\text{Chu-Yang}}{\geq}\sum_{cyc}m_a\Rightarrow\boxed{\sum_{cyc}m_a\leq\sum_{cyc}\frac{w_ah_a}{r_a}}\ \textit{and so,}$$

$$\sum_{cyc}\frac{h_bh_c}{h_a}\leq\sum_{cyc}m_a\leq\sum_{cyc}\frac{w_ah_a}{r_a}\ \forall\,\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equi}\textit{lateral}\ (QED)$$

4368. In ΔABC the following relationship holds:

$$\sum_{cyc}m_a(\sin A-\sin B)\leq 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{WLOG: } a\leq b\leq c\Rightarrow\begin{cases}m_a\geq m_b\geq m_c\\ \sin A\leq\sin B\leq\sin C\end{cases}$$

By rearrangements' inequality:

$$m_a\sin A+m_b\sin B+m_c\sin C\leq m_a\sin B+m_b\sin C+m_c\sin A$$

$$m_a(\sin A-\sin B)+m_b(\sin B-\sin C)+m_c(\sin C-\sin A)\leq 0$$

$$\sum_{cyc}m_a(\sin A-\sin B)\leq 0$$

Equality holds for $a=b=c$.

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4369. In any ΔABC the following relationship holds :

$$\sum_{cyc}\left(\sqrt{\frac{\sin B+\sin C}{\sin A+\sin B}}\cdot\frac{\sqrt{\sin B+\sin C}+\sqrt{\sin A+\sin C}}{\sqrt{(\sin A+\sin B)(\sin B+\sin C)}+\sin A+\sin C}\right)\geq\sqrt{3\sqrt{3}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall A',B',C',x',y',z'>0,$$

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\underbrace{\geq}_{①}\sqrt{3\sum_{cyc}A'B'}$$

$$\text{Now, }\sum_{cyc}\left(\sqrt{\frac{\sin B+\sin C}{\sin A+\sin B}}\cdot\frac{\sqrt{\sin B+\sin C}+\sqrt{\sin A+\sin C}}{\sqrt{(\sin A+\sin B)(\sin B+\sin C)}+\sin A+\sin C}\right)$$

$$=\sum_{cyc}\left(\frac{x}{z}\cdot\frac{x+y}{zx+y^2}\right)\ \left(x=\sqrt{\sin B+\sin C},y=\sqrt{\sin C+\sin A},z=\sqrt{\sin A+\sin B}\right)$$

$$=\sum_{cyc}\left(\frac{x^2y}{y^2z+z^2x}\cdot\frac{x(x+y)}{x^2y}\right)=\sum_{cyc}\left(\frac{x^2y}{y^2z+z^2x}\cdot\left(\frac{1}{x}+\frac{1}{y}\right)\right)$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(x'=x^2y,y'=y^2z,z'=z^2x,A'=\frac{1}{z},B'=\frac{1}{x},C'=\frac{1}{y}\right)\overset{\text{via }①}{\geq}\sqrt{3\sum_{cyc}\left(\frac{1}{xy}\right)}\overset{\text{Bergstrom}}{\geq}$$

$$\sqrt{\frac{27}{\sum_{cyc}xy}}\geq\sqrt{\frac{27}{\sum_{cyc}x^2}}=\sqrt{\frac{27}{2\sum_{cyc}\sin A}}=\sqrt{\frac{27}{\left(\frac{2s}{R}\right)}}\overset{\text{Mitrinovic}}{\geq}\sqrt{\frac{27}{3\sqrt{3}}}=\sqrt{3\sqrt{3}}$$

$$\text{so, }\sum_{cyc}\left(\sqrt{\frac{\sin B+\sin C}{\sin A+\sin B}}\cdot\frac{\sqrt{\sin B+\sin C}+\sqrt{\sin A+\sin C}}{\sqrt{(\sin A+\sin B)(\sin B+\sin C)}+\sin A+\sin C}\right)\geq\sqrt{3\sqrt{3}}$$

$\forall\Delta ABC,''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$

4370. In ΔABC the following relationship holds:

$$\sum_{cyc}w_a\left(\cot\frac{A}{2}-\cot\frac{B}{2}\right)\geq 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{WLOG: } a \le b \le c \Rightarrow \begin{cases} w_a \ge w_b \ge w_c \\ \cot\frac{A}{2} \ge \cot\frac{B}{2} \ge \cot\frac{C}{2} \end{cases}$$

By rearrangements' inequality:

$$w_a \cot\frac{A}{2} + w_b \cot\frac{B}{2} + w_c \cot\frac{C}{2} \ge w_a \cot\frac{B}{2} + w_b \cot\frac{C}{2} + w_c \cot\frac{A}{2}$$

$$w_a\left(\cot\frac{A}{2} - \cot\frac{B}{2}\right) + w_b\left(\cot\frac{B}{2} - \cot\frac{C}{2}\right) + w_c\left(\cot\frac{C}{2} - \cot\frac{A}{2}\right) \ge 0$$

$$\sum_{cyc} w_a\left(\cot\frac{A}{2} - \cot\frac{B}{2}\right) \ge 0$$

Equality holds for $a = b = c$.

4371. In any ΔABC the following relationship holds :

$$\frac{\csc^2\frac{A}{2} + \csc\frac{B}{2}\csc\frac{C}{2}}{\csc\frac{B}{2} + \csc\frac{C}{2}} + \frac{\csc^2\frac{B}{2} + \csc\frac{C}{2}\csc\frac{A}{2}}{\csc\frac{C}{2} + \csc\frac{A}{2}} + \frac{\csc^2\frac{C}{2} + \csc\frac{A}{2}\csc\frac{B}{2}}{\csc\frac{A}{2} + \csc\frac{B}{2}} \ge 6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\csc^2\frac{A}{2} + \csc\frac{B}{2}\csc\frac{C}{2}}{\csc\frac{B}{2} + \csc\frac{C}{2}} = \frac{\frac{1}{\sin^2\frac{A}{2}} + \frac{1}{\sin\frac{B}{2}\sin\frac{C}{2}}}{\frac{1}{\sin\frac{B}{2}} + \frac{1}{\sin\frac{C}{2}}} = \sum_{cyc}\frac{\left(\frac{1}{\sin\frac{A}{2}}\right)^2}{\frac{1}{\sin\frac{B}{2}} + \frac{1}{\sin\frac{C}{2}}} + \sum_{cyc}\frac{1}{\sin\frac{B}{2} + \sin\frac{C}{2}}$$

$$\overset{\text{Bergstrom}}{\ge} \frac{\left(\sum_{cyc}\frac{1}{\sin\frac{A}{2}}\right)^2}{2\sum_{cyc}\frac{1}{\sin\frac{A}{2}}} + \frac{9}{2\sum_{cyc}\sin\frac{A}{2}} \overset{\text{Jensen}}{\ge} \frac{3}{2}.\csc\frac{\pi}{6} + \frac{9}{6\sin\frac{\pi}{6}} = 3 + 3 = 6 \text{ and so,}$$

$$\sum_{cyc}\frac{\csc^2\frac{A}{2} + \csc\frac{B}{2}\csc\frac{C}{2}}{\csc\frac{B}{2} + \csc\frac{C}{2}} \ge 6 \ \forall \Delta ABC, '' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4372. In ΔABC the following relationship holds:

$$\sum_{cyc} h_a(w_a - w_b) \geq 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{WLOG: } a \leq b \leq c \Rightarrow \begin{cases} h_a \geq h_b \geq h_c \\ w_a \geq w_b \geq w_c \end{cases}$$

By rearrangements' inequality: $h_a w_a + h_b w_b + h_c w_c \leq h_a w_b + h_b w_c + h_c w_a$

$$h_a(w_a - w_b) + h_b(w_b - w_c) + h_c(w_c - w_a) \leq 0$$

$$\sum_{cyc} h_a(w_a - w_b) \geq 0$$

Equality holds for $a = b = c$.

4373. In ΔABC the following relationship holds:

$$\sum_{cyc} r_a(\sin A - \sin B) \geq 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{WLOG: } a \leq b \leq c \Rightarrow -a \geq -b \geq -c \Rightarrow s - a \geq s - b \geq s - c$$

$$\Rightarrow \frac{1}{s-a} \leq \frac{1}{s-b} \leq \frac{1}{s-c} \Rightarrow \frac{F}{s-a} \leq \frac{F}{s-b} \leq \frac{F}{s-c} \Rightarrow r_a \leq r_b \leq r_c$$

$$a \leq b \leq c \Rightarrow \sin A \leq \sin B \leq \sin C$$

By rearrangements' inequality:

$$r_a \sin A + r_b \sin B + r_c \sin C \leq r_a \sin B + r_b \sin C + r_c \sin A$$

$$r_a(\sin A - \sin B) + r_b(\sin B - \sin C) + r_c(\sin C - \sin A) \leq 0$$

$$\sum_{cyc} r_a(\sin A - \sin B) \leq 0$$

Equality holds for $a = b = c$.

4374. In ΔABC the following relationship holds:

$$\sum_{cyc} m_a\left(\sin\frac{A}{2}-\sin\frac{B}{2}\right)\le 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{WLOG: } a\le b\le c\Rightarrow\begin{cases} m_a\ge m_b\ge m_c\\ \sin\frac{A}{2}\le\sin\frac{B}{2}\le\sin\frac{C}{2}\end{cases}$$

By rearrangements' inequality:

$$m_a\sin\frac{A}{2}+m_b\sin\frac{B}{2}+m_c\sin\frac{C}{2}\le m_a\sin\frac{B}{2}+m_b\sin\frac{C}{2}+m_c\sin\frac{A}{2}$$

Equality holds for $a=b=c$.

4375. In acute ΔABC the following relationship holds:

$$\sum_{cyc} w_a\left(\tan A-\tan B\right)\le 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{WLOG: } a\le b\le c\Rightarrow\begin{cases} w_a\ge w_b\ge w_c\\ \tan A\le\tan B\le\tan C\end{cases}$$

By rearrangements' inequality:

$$w_a\tan A+w_b\tan B+w_c\tan C\le w_a\tan B+w_b\tan C+w_c\tan A$$

$$w_a(\tan A-\tan B)+w_b(\tan B-\tan C)+w_c(\tan C-\tan A)\le 0$$

$$\sum_{cyc} w_a(\tan A-\tan B)\le 0$$

Equality holds for $a=b=c$.

4376. In ΔABC the following relationship holds:

$$\sum_{cyc} h_a\left(\sin\frac{A}{2}-\sin\frac{B}{2}\right)\le 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{WLOG: } a \le b \le c \Rightarrow \begin{cases} h_a \ge h_b \ge h_c \\ \sin\frac{A}{2} \le \sin\frac{B}{2} \le \sin\frac{C}{2} \end{cases}$$

By rearrangements' inequality:

$$h_a \sin\frac{A}{2} + h_b \sin\frac{B}{2} + h_c \sin\frac{C}{2} \le h_a \sin\frac{B}{2} + h_b \sin\frac{C}{2} + h_c \sin\frac{A}{2}$$

$$h_a\left(\sin\frac{A}{2} - \sin\frac{B}{2}\right) + h_b\left(\sin\frac{B}{2} - \sin\frac{C}{2}\right) + h_c\left(\sin\frac{C}{2} - \sin\frac{A}{2}\right) \le 0$$

$$\sum_{cyc} h_a\left(\sin\frac{A}{2} - \sin\frac{B}{2}\right)$$

Equality holds for $a = b = c$.

4377. In acute ΔABC the following relationship holds:

$$\sum_{cyc} h_a(\cot A - \cot B) \ge 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{WLOG: } a \le b \le c \Rightarrow \begin{cases} h_a \ge h_b \ge h_c \\ \cot A \ge \cot B \ge \cot C \end{cases}$$

By rearrangements' inequality:

$$h_a \cot A + h_b \cot B + h_c \cot C \ge h_a \cot B + h_b \cot C + h_c \cot C$$

$$h_a(\cot A - \cot B) + h_b(\cot B - \cot C) + h_c(\cot C - \cot A) \ge 0$$

$$\sum_{cyc} h_a(\cot A - \cot B) \ge 0$$

Equality holds for $a = b = c$.

4378. In acute ΔABC the following relationship holds:

$$\sum_{cyc} h_a(\tan A - \tan B) \le 0$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{WLOG: } a \le b \le c \Rightarrow \begin{cases} h_a \ge h_b \ge h_c \\ \tan A \le \tan B \le \tan C \end{cases}$$

By rearrangements' inequality:

$$h_a \tan A + h_b \tan B + h_c \tan C \le h_a \tan B + h_b \tan C + h_c \tan A$$

$$h_a(\tan A - \tan B) + h_b(\tan B - \tan C) + h_c(\tan C - \tan A) \le 0$$

$$\sum_{cyc} h_a(\tan A - \tan B) \le 0$$

Equality holds for $a = b = c$.

4379. In ΔABC holds :

$$\sum_{cyc} \frac{h_b h_c}{h_a} \le \sum_{cyc} m_a \le \frac{3}{2} \cdot \sqrt{a^2 + b^2 + c^2} \le \sum_{cyc} \frac{g_a h_a}{r_a} \le \sum_{cyc} \frac{w_a h_a}{r_a}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc} \frac{g_a h_a}{r_a}\right)^2 \overset{\text{Bogdan Fustei}}{=} \sum_{cyc} \left(\frac{(s-a)^4 + 2rh_a(s-a)^2}{r^2 s^2} \cdot \frac{b^2 c^2}{4R^2}\right) +$$

$$2\sum_{cyc} \left(\frac{h_b h_c}{r_b r_c} \cdot \sqrt{((s-b)^2 + 2rh_b)((s-c)^2 + 2rh_c)}\right) \to (*)$$

$$\bullet \sum_{cyc} ((s-a)^4 b^2 c^2) = s^4((s^2 + 4Rr + r^2)^2 - 16Rrs^2) - 16Rrs^4(s^2 + 4Rr + r^2)$$

$$+288R^2r^2s^4 - 128R^2r^2s^4 + 32R^2r^2s^2(s^2 - 4Rr - r^2) \to (**)$$

$$\bullet\, 2r\sum_{cyc} (h_a(s-a)^2 b^2 c^2) =$$

$$\frac{r}{R}\begin{pmatrix} s^2\left((s^2 + 4Rr + r^2)^3 - 24Rrs^2(s^2 + 2Rr + r^2)\right) - \\ 8Rrs^2((s^2 + 4Rr + r^2)^2 - 16Rrs^2) + 16R^2r^2s^2(s^2 + 4Rr + r^2) \end{pmatrix} \to (***)$$

$$\text{Now, } 2\sum_{cyc} \left(\frac{h_b h_c}{r_b r_c} \cdot \sqrt{((s-b)^2 + 2rh_b)((s-c)^2 + 2rh_c)}\right) \overset{\text{Reverse CBS}}{\ge}$$

$$\frac{4r}{R} \cdot \frac{2R}{r} \cdot \sum_{cyc} \left(\sin^2 \frac{A}{2}\left((s-b)(s-c) + \frac{2r}{2R} a \cdot \sqrt{bc}\right)\right)$$

$$\overset{GM-HM}{\ge} \frac{8}{4Rrs} \cdot \sum_{cyc} (a(s-b)^2(s-c)^2) + \frac{8r}{R} \cdot \sum_{cyc} \left(\frac{(s-b)(s-c)}{bc} \cdot \frac{2abc}{b+c}\right)$$

$$=\frac{2}{Rrs}\cdot\left(\begin{matrix}4Rrs(s^2+4Rr+r^2)-32Rrs^3+s^2.2s(s^2-2Rr+r^2)+48Rrs^3-\\4s^3(s^2+4Rr+r^2)+2s^5\end{matrix}\right)+$$

$$-s^2\left(2s(s^2-6Rr-3r^2)+2s(s^2+4Rr+r^2)\right)+$$

$$s\left(2((s^2+4Rr+r^2)^2-16Rrs^2)-16r^2s^2+2(s^4-(4Rr+r^2)^2)\right)+$$

$$\frac{16r}{R}\cdot\frac{4Rrs(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)}$$

$$=\frac{4r(8R^2+2Rr-s^2)}{R}+\frac{16r}{R}\cdot\frac{4rs\left((2R-2r)s^2+Rr(4R+r)\right)}{2s(s^2+2Rr+r^2)}$$

$$\therefore 2\sum_{cyc}\left(\frac{h_bh_c}{r_br_c}\cdot\sqrt{((s-b)^2+2rh_b)((s-c)^2+2rh_c)}\right)\geq$$

$$\frac{4r(8R^2+2Rr-s^2)}{R}+\frac{32r^2\left((2R-2r)s^2+Rr(4R+r)\right)}{R(s^2+2Rr+r^2)}\rightarrow(****)$$

So, $(*)$ combined with $(**)+(***)+(****)\Rightarrow\left(\sum_{cyc}\frac{g_ah_a}{r_a}\right)^2\overset{?}{\geq}$

$$\frac{9}{4}.2(s^2-4Rr-r^2)\Leftrightarrow(R+r)s^8-r(22R^2+15Rr-4r^2)s^6+$$

$$r^2(78R^3-4R^2r-27Rr^2+6r^3)s^4+$$

$$r^3(324R^4+512R^3r-230R^2r^2-5Rr^3+4r^4)s^2+$$

$$r^4\left(144R^5+620R^4r+146R^3r^2+8R^2r^3+6Rr^4+r^5\right)\overset{?}{\geq}\boxed{①}\ 0\ and\ \because P=$$

$$(R+r)(s^2-16Rr+5r^2)^4+r(42R^2+29Rr-16r^2)(s^2-16Rr+5r^2)^3$$

$$\overset{Gerretsen}{\geq}\ 0\ \therefore in\ order\ to\ prove\ ①,it\ suffices\ to\ prove:LHS\ of\ ①\overset{?}{\geq}P$$

$$\Leftrightarrow(279R^3+91R^2r-210Rr^2+48r^3)s^4-$$

$$r(7774R^4+288R^3r-6134R^2r^2+2780Rr^3-352r^4)s^2+$$

$$r^2\left(53320R^5-12746R^4r-41415R^3r^2+30299R^2r^3-7722Rr^4+688r^5\right)\overset{?}{\geq}\boxed{②}\ 0\ \&$$

$$\because Q=(279R^3+91R^2r-210Rr^2+48r^3)(s^2-16Rr+5r^2)^2+$$

$$2r(577R^4-83R^3r-748R^2r^2+428Rr^3-64r^4)(s^2-16Rr+5r^2)\overset{Gerretsen}{\geq}0$$

$$\therefore in\ order\ to\ prove\ ②,it\ suffices\ to\ prove:LHS\ of\ ②\overset{?}{\geq}Q$$

$$\Leftrightarrow 90t^5+43t^4-794t^3+828t^2-280t+32\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(90t^4+49t^3+174t^2(t-2)+132t-16)\overset{?}{\geq}0\rightarrow true\ \because t\overset{Euler}{\geq}2$$

$$\Rightarrow②\Rightarrow①\ is\ true\ \therefore\frac{3}{2}.\sqrt{a^2+b^2+c^2}\leq\sum_{cyc}\frac{g_ah_a}{r_a};Again,triangle\ inequality\Rightarrow$$

$$g_a \le AI + r \overset{?}{\le} w_a \Leftrightarrow \frac{r}{\sin\frac{A}{2}} + r \overset{?}{\le} \frac{8Rrs\cos\frac{A}{2}}{4R\sin\frac{A}{2}\cos\frac{A}{2}(b+c)} \Leftrightarrow \frac{1}{\sin\frac{A}{2}} + 1 \overset{?}{\le} \frac{a+b+c}{(b+c)\sin\frac{A}{2}}$$

$$\Leftrightarrow (b+c)\sin\frac{A}{2} \overset{?}{\le} a \Leftrightarrow \cos\frac{B-C}{2} \overset{?}{\le} 1 \to \text{true} \therefore g_a \le w_a \text{ and analogs}$$

$$\therefore \sum_{cyc}\frac{g_a h_a}{r_a} \le \sum_{cyc}\frac{w_a h_a}{r_a} \text{ and also, } \sum_{cyc} m_a \overset{CBS}{\le} \sqrt{3.\frac{3}{4}\sum_{cyc} a^2} = \frac{3}{2}.\sqrt{\sum_{cyc} a^2} \text{ and}$$

$$\sum_{cyc}\frac{h_b h_c}{h_a} \le \sum_{cyc} m_a \left(\begin{array}{c}\text{Reference : Inequality in Triangle by Dang Ngoc Minh}\\ -397\text{; published at www. ssmrmh. ro}\end{array}\right)$$

$$\forall \Delta ABC, '' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4380. In any ΔABC the following relationship holds :

$$w_a^2 r_a + w_b^2 r_b + w_c^2 r_c \le m_a w_b w_c + m_b w_c w_a + m_c w_a w_b$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}(w_a^2 r_a) = \sum_{cyc}\left(\frac{4b^2c^2}{(b+c)^2}\cos^2\frac{A}{2}.s\tan\frac{A}{2}\right) = \sum_{cyc}\left(\frac{4b^2c^2}{(b+c)^2}.\frac{as}{4R}\right)$$

$$= 4rs^2.\frac{1}{4s^2(s^2+2Rr+r^2)^2}.\sum_{cyc}\left(bc\left(a^4+\left(\sum_{cyc}ab\right)^2+2a^2\left(\sum_{cyc}ab\right)\right)\right)$$

$$= \frac{r\left(\begin{array}{c}8Rrs^2(s^2-6Rr-3r^2)+(s^2+4Rr+r^2)^3+\\ 16Rrs^2(s^2+4Rr+r^2)\end{array}\right)}{(s^2+2Rr+r^2)^2} = \sum_{cyc}(w_a^2 r_a) \to (1)$$

$$\text{Again, } \sum_{cyc}(m_a w_b w_c) \overset{\text{Gerretsen}}{\ge} \frac{16Rr^2s^2}{s^2+2Rr+r^2}.\sum_{cyc}\frac{m_a}{w_a} \overset{\text{Lascu}}{\ge} \frac{16Rr^2s^2}{s^2+2Rr+r^2}.\sum_{cyc}\frac{\frac{b+c}{2}.\cos\frac{A}{2}}{\frac{2bc}{b+c}..\cos\frac{A}{2}}$$

$$= \frac{16Rr^2s^2}{s^2+2Rr+r^2}.\sum_{cyc}\frac{a(b+c)^2}{4abc} =$$

$$\frac{16Rr^2s^2}{s^2+2Rr+r^2}\cdot\frac{1}{16Rrs}\cdot\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+3abc\right)$$

$$=\frac{16Rr^2s^2}{s^2+2Rr+r^2}\cdot\frac{2s}{16Rrs}\cdot(s^2+10Rr+r^2)\overset{?}{\geq}\sum_{cyc}(w_a^2r_a)\overset{via\ (1)}{=}$$

$$\frac{r\left(8Rrs^2(s^2-6Rr-3r^2)+(s^2+4Rr+r^2)^3+16Rrs^2(s^2+4Rr+r^2)\right)}{(s^2+2Rr+r^2)^2}$$

$$\Leftrightarrow s^6-(12Rr-r^2)s^4-r^2(24R^2-8Rr+r^2)s^2-r^3(4R+r)^3\overset{?}{\geq}\boxed{(1)}\ 0\ and\ \because$$

$$s^4(s^2-16Rr+5r^2)\overset{Gerretsen}{\geq}0\ \therefore\ in\ order\ to\ prove\ (1),\ it\ suffices\ to\ prove:$$

$$LHS\ of\ (1)\overset{?}{\geq}s^4(s^2-16Rr+5r^2)\Leftrightarrow(4R-4r)s^4-r(24R^2-8Rr+r^2)s^2-$$

$$r^2(4R+r)^3\overset{?}{\geq}\boxed{(2)}\ 0\ and\ \because\ s^2(s^2-16Rr+5r^2)\overset{Gerretsen}{\geq}0\ \therefore\ in\ order\ to\ prove\ (2),$$

$$it\ suffices\ to\ prove:\ LHS\ of\ (2)\overset{?}{\geq}s^2(s^2-16Rr+5r^2)$$

$$\Leftrightarrow(40R^2-76Rr+19r^2)s^2\overset{?}{\geq}\boxed{(3)}\ r(4R+r)^3\ and\ finally,$$

$$(40R^2-76Rr+19r^2)s^2\overset{Gerretsen}{\geq}(40R^2-76Rr+19r^2)(16Rr-5r^2)$$

$$\overset{?}{\geq}r(4R+r)^3\Leftrightarrow 24t^3-61t^2+28t-4\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(24t^2-13t+2)\overset{?}{\geq}0\rightarrow true\ \because\ t\overset{Euler}{\geq}2\Rightarrow(3)\Rightarrow(2)\Rightarrow(1)\ is\ true$$

$$\therefore w_a^2r_a+w_b^2r_b+w_c^2r_c\leq m_aw_bw_c+m_bw_cw_a+m_cw_aw_b\ \forall\ \Delta ABC,$$

$$''=''\ iff\ \Delta ABC\ is\ equilateral\ (QED)$$

4381. In any ΔABC the following relationship holds:

$$\frac{9}{4}a^2+h_a^2\geq 2\left(h_b^2+h_c^2\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x=s-a, y=s-b, z=s-c$ and then : $a=y+z, b=z+x,$
$c=x+y$ and $s=x+y+z$ and furthermore, we denote : $\frac{y+z}{x}=m$ and $\frac{yz}{x^2}=n$
and then, we have the following set "S" of relations : $y^2+z^2=x^2(m^2-2n)$,
$y^3+z^3=x^3(m^3-3nn), y^4+z^4=x^4((m^2-2n)^2-2n^2)$,
$y^5+z^5=x^5\left(m((m^2-2n)^2+n^2-nm^2)\right), y^6+z^6=x^6((m^3-3nn)^2-2n^3)$,
and now, $\frac{9}{4}a^2+h_a^2\overset{?}{\geq}2\left(h_b^2+h_c^2\right)\Leftrightarrow 9R^2a^2+b^2c^2\overset{?}{\geq}2(c^2a^2+a^2b^2)\Leftrightarrow$
$9a^4b^2c^2+16s(s-a)(s-b)(s-c)b^2c^2\overset{?}{\geq}32s(s-a)(s-b)(s-c)a^2(b^2+c^2)$
$\Leftrightarrow 9(z+x)^2(x+y)^2(y+z)^4+16xyz(x+y+z)(z+x)^2(x+y)^2$
$\overset{?}{\geq}32xyz(x+y+z)(y+z)^2((z+x)^2+(x+y)^2)$
$\Leftrightarrow 16x^6.yz+48x^5.yz(y+z)+9x^4(y^4+z^4)+20x^4.yz(y^2+z^2)+54x^4.y^2z^2+$
$18x^3\left(y^5+z^5\right)-22x^3.yz(y^3+z^3)-92x^3.y^2z^2(y+z)+9x^2(y^6+z^6)-$
$24x^2.yz(y^4+z^4)-81x^2.y^2z^2(y^2+z^2)-80x^2.y^3z^3-14x.yz\left(y^5+z^5\right)-$
$6x.y^2z^2(y^3+z^3)+68x.y^3z^3(y+z)+9y^2z^2(y+z)^4\overset{?}{\geq}0$ (①) and via
set of relations "S", to prove ①, suffices to prove, following simplification :
$(9m^4+64m^3+96m^2+64m+32)n^2-$
$\left(14m^5+78m^4+112m^3+16m^2-48m-16\right)n+9m^4(m+1)^2+$
$16(m+1)n^3\overset{?}{\geq}0$ (②) and $\because n(m+1)(4n-m^2)^2\geq 0\therefore$ in order to prove ②,
it suffices to prove : LHS of ② $\overset{?}{\geq}n(m+1)(4n-m^2)^2$
$\Leftrightarrow \overbrace{(9m^4+72m^3+104m^2+64m+32)}^{\Omega_1}n^2-$
$\overbrace{(15m^5+79m^4+112m^3+16m^2-48m-16)}^{\Omega_2}n+\overbrace{9m^4(m+1)^2}^{\Omega_3}+\overset{?}{\geq}0$ (③)
and it's trivially true when : $\Omega_2\leq 0$ and when : $\Omega_2>0$, then : $\Omega_2=$
$\left(3m^4+\frac{88m^3}{5}+\frac{824m^2}{25}+\frac{2872m}{125}+\frac{2616}{625}\right)(5m-3)-\frac{2152}{625}>0\Rightarrow$
$\left(3m^4+\frac{88m^3}{5}+\frac{824m^2}{25}+\frac{2872m}{125}+\frac{2616}{625}\right)(5m-3)>\frac{2152}{625}>0\Rightarrow m>\frac{3}{5}\rightarrow(*)$
Now, LHS of ③ is a quadratic polynomial in "n" with discriminant,
$\boxed{\delta}=\Omega_2^2-4\Omega_1\Omega_3=$
$-(m+1)^2(3m+2)(m-2)^2\left(33m^5+334m^4+472m^3+80m^2-112m-32\right)$

$$= -(m+1)^2(3m+2)(m-2)^2\left(\left(\frac{33m^4}{5}+\frac{1769m^3}{25}+\frac{17107m^2}{125}+\frac{61321m}{625}+36+\frac{1463}{3125}\right)(5m-3)+77+\frac{1264}{3125}\right)$$

$\overset{via\ (*)}{\leq} 0$ *and so,* ③ ⇒ ② ⇒ ① *is true* $\therefore \frac{9}{4}a^2 + h_a^2 \geq 2\left(h_b^2 + h_c^2\right) \forall\, \Delta$ ABC, $''='' \Leftrightarrow \Delta$ ABC *is equilateral* (QED)

4382. In any Δ ABC holds :

$$g_b g_c \leq \min\left\{w_a r_a, \frac{\sqrt{(4R+19r)r}}{2}a\right\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x = s-a, y = s-b, z = s-c$ and then : $a = y+z, b = z+x,$
$c = x+y$ and $s = x+y+z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$
and then, we have the following set $''S''$ of relations : $y^2+z^2 = x^2(m^2-2n)$,

$$\overset{squaring\ and\ via\ Bogdan\ Fustei}{\Leftrightarrow} y^3+z^3 = x^3(m^3-3nn), \text{and now, } g_b g_c \overset{?}{\leq} r_a w_a$$

$$\left((s-b)^2+\frac{4(s-a)(s-b)(s-c)}{b}\right)\left((s-c)^2+\frac{4(s-a)(s-b)(s-c)}{c}\right) \overset{?}{\leq} \frac{s(s-b)(s-c)}{s-a}\cdot\frac{4bc}{(b+c)^2}\cdot s(s-a)$$

$$\Leftrightarrow 4(z+x)^2(x+y)^2(x+y+z)^2 \overset{?}{\geq} (4zx+xy+yz)(4xy+zx+yz)(2x+y+z)^2$$

$$\Leftrightarrow 4x^6+16x^5(y+z)+8x^4(y^2+z^2)-12x^4.yz-32x^3.yz(y+z)-$$

$$5x^2.yz(y+z)^2+3x.yz(y^3+z^3)+13x.y^2z^2(y+z)+3y^2z^2(y+z)^2 \overset{?}{\geq} 0 \quad (①)$$

and via set of relations $''S''$, to prove ①, it suffices to prove, following simplification :

$$\overbrace{(3m^2+4m)}^{\Omega_1}n^2+\overbrace{(3m^3-5m^2-32m-28)}^{\Omega_2}n+\overbrace{(8m^2+16m+4)}^{\Omega_3}+\overset{?}{\geq} 0 \quad (②)$$

and it's trivially true when : $\Omega_2 \geq 0$ *and* when : $\Omega_2 < 0$, then :
$3m^3-5m^2-32m-28 = (m-5)(3m^2+10m+18)+62 < 0$
$\Rightarrow (m-5)(3m^2+10m+18) < -62 < 0 \Rightarrow m < 5 \rightarrow$ (i)
Now, LHS of ② is *a quadratic polynomial* in $''n''$ with discriminant, $\boxed{\delta} =$

$\sigma_2^2-4\sigma_1\sigma_3=(m+1)(m+2)^2(9m^3-75m^2+40m+196)$ and when :
$9m^3-75m^2+40m+196\le 0$, then $\delta\le 0\Rightarrow$ (2) is trivially true and when :
$9m^3-75m^2+40m+196>0$, then : $9m^3-75m^2+40m+196$

$$=\left(3m(m-5)-3m-\frac{86}{3}\right)(3m-7)-\frac{14}{3}>0$$

$$\Rightarrow\left(3m(m-5)-3m-\frac{86}{3}\right)(3m-7)>\frac{14}{3}>0 \text{ and } \because m\overset{\text{via (i)}}{\le}5$$

$$\Rightarrow 3m(m-5)-3m-\frac{86}{3}<0\therefore m<\frac{7}{3}\to\text{(ii)}$$ and so, when : $\delta>0$,

then it suffices to prove : $2\Omega_1.n\overset{?}{\le}-\Omega_2-\sqrt{\delta}\Leftrightarrow$ and $\because n\overset{AM-GM}{\le}\frac{m^2}{4}$

$$\therefore \text{suffices to prove : } \Omega_1.\frac{m^2}{2}\overset{?}{\le}-\Omega_2-\sqrt{\delta}\Leftrightarrow -(m+2)^2(3m^2-2m-14)\overset{?}{\ge}2.\sqrt{\delta}$$

and $\because 3m^2-2m-14=\left(m+\frac{5}{3}\right)(3m-7)-\frac{7}{3}\overset{\text{via (ii)}}{<}0\therefore$ it suffices to prove :

$$(m+2)^4(3m^2-2m-14)^2\overset{?}{\ge}4(m+1)(m+2)^2(9m^3-75m^2+40m+196)$$

$$\Leftrightarrow \boxed{m(m+2)^2(m-2)^2(3m+4)(3m^2+16m+4)\overset{?}{\ge}0}\therefore g_bg_c\le r_aw_a \text{ and now,}$$

$$g_bg_c\overset{?}{\le}\frac{\sqrt{(4R+19r)r}}{2}a\Leftrightarrow\left(\frac{(s-b)^2+4(s-a)(s-b)(s-c)}{b}\right)\left(\frac{(s-c)^2+4(s-a)(s-b)(s-c)}{c}\right)$$

$$\overset{?}{\le}\frac{\big(abc+19(s-a)(s-b)(s-c)\big)^2}{4}.a^2\Leftrightarrow$$

$$\big((y+z)(z+x)(x+y)+19xyz\big)(y+z)^2(z+x)(x+y)\overset{?}{\ge}$$
$$4(x+y+z)yz(y(z+x)+4zx)(z(x+y)+4xy)\Leftrightarrow$$

$$(m^3-m^2-56m-36)n^2+(2m^4+5m^3+3m^2)n+m^3(m+1)^2\overset{?}{\underset{(*)}{\ge}}4mn^3+4n^3$$

and $\because 4mn^3+4n^3\le 4mn^2.\frac{m^2}{4}+4n^2.\frac{m^2}{4}\therefore$ to prove $(*)$, it suffices to prove :

LHS of $(*)\overset{?}{\ge}m^3n^2+m^2n^2\Leftrightarrow\overbrace{(2m^2+56m+36)}^{\sigma_1}n^2-\overbrace{(2m^4+5m^3+3m^2)}^{\sigma_2}n-\overbrace{m^3(m+1)^2}^{\sigma_3}\overset{?}{\underset{(**)}{\ge}}0$; discriminant, δ_0 of LHS of $(**)=(2m^4+5m^3+3m^2)^2+4(2m^2+56m+36)m^3(m+1)^2>0\therefore$ to prove $(**)$, suffices to prove :

$$2\sigma_1.n\overset{?}{\underset{(i)}{\ge}}\sigma_2+\sqrt{\delta_0}\ \&\ 2\sigma_1.n\overset{?}{\underset{(ii)}{\ge}}\sigma_2-\sqrt{\delta_0};\ \text{now, } 2\sigma_1.n\overset{AM-GM}{\le}2\sigma_1.\frac{m^2}{4}\overset{?}{\le}\sigma_2+\sqrt{\delta_0}$$

$$\Leftrightarrow 2m^2(-m^2+23m+15)\overset{?}{\le}2\sqrt{\delta_0},\ \text{trivially true if : } -m^2+23m+15\le 0$$

and when : $-m^2+23m+15>0$, *then* : $2m^2(-m^2+23m+15)\overset{?}{\le}2\sqrt{\delta_0}\Leftrightarrow$

$$\delta_0 \overset{?}{\geq} m^4(-m^2+23m+15)^2 \Leftrightarrow m^3(3m+2)(m-2)^2(m^2+28m+18) \overset{?}{\geq} 0$$

$\rightarrow$ true $\because m > 0 \Rightarrow$ (i) is true; also, $2\sigma_1 . n + \sqrt{\delta_0} > \sqrt{\delta_0} > \sqrt{(2m^4+5m^3+3m^2)^2}$

$= \sigma_2 \Rightarrow$ (ii) is trueSo, (i), (ii) both are true $\Rightarrow (**) \Rightarrow (*)$ is true

$$\therefore g_b g_c \leq \frac{\sqrt{(4R+19r)r}}{2} a \text{ and so, } g_b g_c \leq \min\left\{w_a r_a, \frac{\sqrt{(4R+19r)r}}{2} a\right\} \forall \Delta ABC,$$

$''='' $ iff Δ ABC is equilateral (QED)

4383. In any Δ ABC the following relationship holds :

$$\frac{w_a^2}{r_a} + \frac{w_b^2}{r_b} + \frac{w_c^2}{r_c} \geq \frac{m_b m_c}{h_a} + \frac{m_c m_a}{h_b} + \frac{m_a m_b}{h_c}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{w_a^2}{r_a} = \sum_{cyc} \left(\frac{4bc}{(b+c)^2} \frac{s(s-a)^2}{rs} \right) =$$

$$= \sum_{cyc} \left(\frac{bc}{r} \cdot \frac{(b+c)^2 + a^2 - 2a(b+c)}{(b+c)^2} \right) =$$

$$= \frac{s^2+4Rr+r^2}{r} +$$

$$\frac{4Rs}{4s^2(s^2+2Rr+r^2)^2} \sum_{cyc} \left(a \left(a^4 + \left(\sum_{cyc} ab \right)^2 + 2a^2 \left(\sum_{cyc} ab \right) \right) \right) -$$

$$\frac{8Rs(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)} = \frac{s^2+4Rr+r^2}{r} +$$

$$\frac{R}{s(s^2+2Rr+r^2)^2} \cdot \begin{pmatrix} 4s(s^2-4Rr-r^2)(s^2-6Rr-3r^2) + 4Rrs(s^2+4Rr+r^2) - \\ 2s\left((s^2+4Rr+r^2)^2 - 16Rrs^2\right) \\ +2s(s^2+4Rr+r^2)^2 + 4s(s^2+4Rr+r^2)(s^2-6Rr-3r^2) \end{pmatrix}$$

$$-\frac{4R(5s^2+4Rr+r^2)}{s^2+2Rr+r^2} \overset{?}{\geq} \sum_{cyc} \frac{a(2a^2+bc)}{8rs} = \frac{4s(s^2-6Rr-3r^2)+12Rrs}{8rs} \Leftrightarrow$$

$$s^6 - (9Rr-7r^2)s^4 - r^2(88R^2+50Rr-11r^2)s^2 +$$

$$r^3(12R^3+24R^2r+23Rr^2+5r^3) \boxed{\overset{?}{\geq} 0} \text{ and } \because s^4(s^2-16Rr+5r^2) \overset{\text{Gerretsen}}{\geq} 0$$

$\therefore$ in order to prove ①, it suffices to prove : LHS of ① $\overset{?}{\geq} s^4(s^2-16Rr+5r^2)$

$$\Leftrightarrow (7R+2r)s^4-r(88R^2+50Rr-11r^2)s^2+$$
$$r^3(12R^3+24R^2r+23Rr^2+5r^3)\overset{?}{\geq}0 \text{ and } \because s^2(s^2-16Rr+5r^2)\overset{\text{Gerretsen}}{\geq}0$$

$$\therefore \text{in order to prove } (2), \text{it suffices to prove : LHS of } (2)\overset{?}{\geq}$$
$$(7R+2r)s^2(s^2-16Rr+5r^2)\Leftrightarrow(24R^2-53Rr+r^2)s^2+$$
$$r(12R^3+24R^2r+23Rr^2+5r^3)\overset{?}{\geq}0 \text{ and it is trivially true if :}$$
$$24R^2-53Rr+r^2\geq 0 \text{ and when : } 24R^2-53Rr+r^2<0, \text{then :}$$
$$\text{LHS of } (3)\overset{\text{Gerretsen}}{\geq}$$
$$(24R^2-53Rr+r^2)(4R^2+4Rr+3r^2)+r(12R^3+24R^2r+23Rr^2+5r^3)\overset{?}{\geq}0$$
$$\Leftrightarrow 24t^4-26t^3-28t^2-33t+2\overset{?}{\geq}0\left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)(24t^3+22t^2+16t-1)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(3)\Rightarrow(2)\Rightarrow(1)\text{ is true}$$
$$\therefore\sum_{cyc}\frac{w_a^2}{r_a}\geq\sum_{cyc}\frac{a(2a^2+bc)}{8rs}\geq\sum_{cyc}\frac{m_bm_c}{h_a}\left(\because m_bm_c\leq\frac{2a^2+bc}{4}\text{ and analogs}\right)$$
$$\therefore\frac{w_a^2}{r_a}+\frac{w_b^2}{r_b}+\frac{w_c^2}{r_c}\geq\frac{m_bm_c}{h_a}+\frac{m_cm_a}{h_b}+\frac{m_am_b}{h_c}\ \forall\,\Delta ABC,$$
$$''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

4384. In any ΔABC the following relationship holds :

$$(m_a+m_b+m_c)^2+(w_a+w_b+w_c)^2\leq 6\left(w_a^2+w_b^2+w_c^2\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{4s(s-a)}{(b+c)^2}=\sum_{cyc}\frac{(b+c)^2-a^2}{(b+c)^2}=3-\sum_{cyc}\frac{(2s-(2s-a)^2)}{(2s-a)^2}$$
$$=4s\sum_{cyc}\frac{1}{b+c}-4s^2\left(\left(\sum_{cyc}\frac{1}{b+c}\right)^2-\frac{2}{2s(s^2+2Rr+r^2)}\cdot\sum_{cyc}(b+c)\right)$$
$$=\frac{4s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)}-\frac{(5s^2+4Rr+r^2)^2}{(s^2+2Rr+r^2)^2}+\frac{16s^2}{s^2+2Rr+r^2}\Rightarrow\sum_{cyc}\frac{4s(s-a)}{(b+c)^2}$$
$$=\frac{s^4+(20Rr+18r^2)s^2+r^3(4R+r)}{(s^2+2Rr+r^2)^2}\therefore\sum_{cyc}w_a=\frac{\sqrt{bc}}{b+c}\cdot\sqrt{4s(s-a)}\overset{CBS}{\leq}$$

$$\sqrt{\sum_{cyc} bc}\cdot\sqrt{\sum_{cyc}\frac{4s(s-a)}{(b+c)^2}}=\sqrt{(s^2+4Rr+r^2)\cdot\frac{s^4+(20Rr+18r^2)s^2+r^3(4R+r)}{(s^2+2Rr+r^2)^2}}$$

$$\therefore\left(\sum_{cyc} m_a\right)^2+\left(\sum_{cyc} w_a\right)^2\overset{\text{Chu-Yang}}{\leq}4s^2-16Rr+5r^2+$$

$$(s^2+4Rr+r^2)\cdot\frac{s^4+(20Rr+18r^2)s^2+r^3(4R+r)}{(s^2+2Rr+r^2)^2}\overset{?}{\leq}6\left(\sum_{cyc} w_a^2\right)$$

$$=6\cdot\frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(16R^2+8Rr+r^2)}{(s^2+2Rr+r^2)^2}$$

(Reference : Solution to Inequality in Triangle by Dang Ngoc Minh − 112; published at www. ssmrmh. ro)

$$\Leftrightarrow s^6-(24Rr+14r^2)s^4+r^2(160R^2+140Rr-15r^2)s^2+$$

$$Rr^3(64R^2+124Rr+36r^2)\overset{?}{\geq}\boxed{\forall}\boxed{①}\ 0\ \text{and}\ \because(s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\geq}0$$

$$\therefore\ \text{in order to prove } ①,\ \text{it suffices to prove : LHS of } ①\overset{?}{\geq}(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow(24R-29r)s^4-r(608R^2-620Rr+90r^2)s^2+$$

$$r^2(4160R^3-3716R^2r+1236Rr^2-125r^3)\overset{?}{\geq}\boxed{\forall}\boxed{②}\ 0\ \text{and}\ \because$$

$$(24R-29r)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen and Euler}}{\geq}0\ \therefore\ \text{in order to prove } ②,$$

$$\text{it suffices to prove : LHS of } ②\overset{?}{\geq}(24R-29r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow(40R^2-137Rr+50r^2)s^2\overset{?}{\geq}\boxed{\forall}\boxed{③}\ r(496R^3-1887R^2r+1001Rr^2-150r^3)$$

Case 1 $40R^2-137Rr+50r^2\geq 0$ and then : LHS of ③ $\overset{\text{Gerretsen}}{\geq}$

$$(40R^2-137Rr+50r^2)(16Rr-5r^2)\overset{?}{\geq}\text{RHS of } ③$$

$$\Leftrightarrow 144t^3-505t^2+484t-100\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(35t^2+109t(t-2)+t+50)\overset{?}{\geq}0\to\text{true}\ \because\ t\overset{\text{Euler}}{\geq}2\Rightarrow ③\ \text{is true}$$

Case 2 $40R^2-137Rr+50r^2<0$ and then : LHS of ③ $\overset{\text{Rouche}}{\geq}$

$$(40R^2-137Rr+50r^2)\left(2R^2+10Rr-r^2+2(R-2r)\cdot\sqrt{R^2-2Rr}\right)\overset{?}{\geq}\text{RHS of } ③$$

$$\Leftrightarrow(R-2r)(80R^3-210R^2r+157Rr^2-50r^3)\overset{?}{\geq}$$

$$2(R-2r)\cdot\sqrt{R^2-2Rr}\cdot\left(-(40R^2-137Rr+50r^2)\right)$$

$$\Leftrightarrow 80R^3-210R^2r+157Rr^2-50r^3\overset{?}{\geq}$$
$$2.\sqrt{R^2-2Rr}.\left(-(40R^2-137Rr+50r^2)\right)\left(\because R\overset{Euler}{\geq}2r\right)$$
$$\Leftrightarrow (80R^3-210R^2r+157Rr^2-50r^3)^2\overset{?}{\geq}4(R^2-2Rr)(40R^2-137Rr+50r^2)^2$$
$$\left(\because 80R^3-210R^2r+157Rr^2-50r^3=\right.$$
$$\left.(R-2r)(55R^2+25R(R-2r)+57r^2)+64r^3>0\right)$$
$$\Leftrightarrow (t-2)\left((t-2)(14352t^3+8688t^2(t-2)+1348t+945)+2688\right)+4096\overset{?}{\geq}0$$
$$\rightarrow \text{true (strict inequality)}\because t\overset{Euler}{\geq}2\Rightarrow \textcircled{3}\text{ is true}\therefore \text{combining both cases,}$$
$$\textcircled{3}\Rightarrow\textcircled{2}\Rightarrow\textcircled{1}\text{ is true and so, }(m_a+m_b+m_c)^2+(w_a+w_b+w_c)^2\leq$$
$$6\left(w_a^2+w_b^2+w_c^2\right)\forall\,\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

4385. ***In any ΔABC the following relationship holds:***

$$g_a^2=3r^2+\frac{r^2(R+r)}{R\sin^2\frac{A}{2}}=2(R+r)h_a-r(4R+r)=\frac{(h_a-2r)(4r_a+r_b+r_c)}{2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Jenish Rijal-Bardiya-Nepal

Here, by Stewart's theorem:

$$g_a^2=(s-a)^2+\frac{4(s-a)(s-b)(s-c)}{a}=(s-a)^2+\frac{4\Delta^2}{as}.$$

Here, $\frac{\Delta}{s}=r$ ***and*** $\frac{2\Delta}{a}=h_a$, ***so we have:***

$$\boxed{g_a^2=(s-a)^2+2rh_a.}$$

Via $2Rh_a=bc, 4Rr=\frac{abc}{s}$, ***and*** $r^2=\frac{(s-a)(s-b)(s-c)}{s}$, ***we have:***

$$2(R+r)h_a-r(4R+r)=2Rh_a-4Rr-r^2+2rh_a=bc-\frac{abc}{s}-r^2+2rh_a=$$
$$=\frac{s-a}{s}[bc-(s-b)(s-c)]+2rh_a=\frac{s-a}{s}[s(s-a)]+2rh_a=(s-a)^2+2rh_a$$
$$=g_a^2$$

$$\therefore\ \boxed{2(R+r)h_a-r(4R+r)=g_a^2.}$$

Via $\sin^2\frac{A}{2}=\frac{(s-b)(s-c)}{bc}$ ***and*** $bc=2Rh_a$:

$$R\sin^2\frac{A}{2}=\frac{(s-b)(s-c)}{2h_a}\Rightarrow\frac{r^2(R+r)}{R\sin^2\frac{A}{2}}=2(R+r)h_a\left(\frac{r^2}{(s-b)(s-c)}\right)$$

$As\ \frac{r^2}{(s-b)(s-c)}=\frac{s-a}{s}=1-\frac{a}{s}\ \ and\ \ ah_a=2rs, we\ get:$

$$3r^2+2(R+r)h_a\left(1-\frac{a}{s}\right)=3r^2+2(R+r)h_a-\frac{2(R+r)(2rs)}{s}=$$

$$=3r^2+2(R+r)h_a-4r(R+r)\ =2(R+r)h_a-r(4R+r).$$

$$\therefore\ \boxed{3r^2+\frac{r^2(R+r)}{R\sin^2\frac{A}{2}}=2(R+r)h_a-r(4R+r).}$$

$At\ last, via\ r_a+r_b+r_c=4R+r\ and\ r_a(h_a-2r)=rh_a, we\ have$

$$\frac{(h_a-2r)(4r_a+r_b+r_c)}{2}=\frac{(h_a-2r)(3r_a+4R+r)}{2}=$$

$$=\frac{3}{2}r_a(h_a-2r)+\frac{1}{2}(h_a-2r)(4R+r)$$

$$=\frac{3}{2}rh_a+\frac{1}{2}h_a(4R+r)-r(4R+r)$$

$$=2(R+r)h_a-r(4R+r)$$

$$\therefore\ \boxed{\frac{(h_a-2r)(4r_a+r_b+r_c)}{2}=2(R+r)h_a-r(4R+r).}$$

4386. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\sqrt{\frac{1}{2}\left(\frac{n_bn_c}{r_br_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}-1\right)}\le\sqrt{\frac{max(r_a,r_b,r_c)+AI+BI+CI}{r}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt{\frac{1}{2}\left(\frac{n_bn_c}{r_br_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}-1\right)}=\sum_{cyc}\sqrt{\frac{1}{2}\left(\frac{n_bn_c}{r_br_c}+\frac{2r_bh_b+2r_ch_c}{2r_br_c}-1\right)}$$

$$\overset{\text{Bogdan Fustei}}{=}\sum_{cyc}\sqrt{\frac{1}{2}\left(\frac{2n_bn_c}{2r_br_c}+\frac{s^2-n_b^2+s^2-n_c^2}{2r_br_c}-1\right)}$$

$$=\sum_{cyc}\sqrt{\frac{1}{2}\left(\frac{2s^2-(n_b-n_c)^2}{2r_br_c}-1\right)}\le\sum_{cyc}\sqrt{\frac{1}{2}\left(\frac{s^2-s(s-a)}{s(s-a)}\right)}$$

$$=\sum_{cyc}\sqrt{\frac{1}{2}\cdot\frac{4R\cos\frac{A}{2}\sin\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}}=\sum_{cyc}\sqrt{\frac{2R}{r}\cdot\sin^2\frac{A}{2}}$$

$$=\sqrt{\frac{2R}{r}}\cdot\sqrt{\sum_{cyc}\sin^2\frac{A}{2}+2\sum_{cyc}\left(\sin\frac{B}{2}\sin\frac{C}{2}\right)}=\sqrt{\frac{2R}{r}}\cdot\sqrt{\frac{2R-r}{2R}+\frac{1}{2R}\cdot\sum_{cyc}\frac{r}{\sin\frac{A}{2}}}$$

$$=\sqrt{\frac{2R-r}{r}+\frac{1}{r}\sum_{cyc}AI}\ \therefore\ \sum_{cyc}\sqrt{\frac{1}{2}\left(\frac{n_bn_c}{r_br_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}-1\right)}\le\sqrt{\frac{2R-r}{r}+\frac{1}{r}\sum_{cyc}AI}\to(1)$$

$$\text{Now, } max(r_a,r_b,r_c)\ge\sqrt{4R^2-7r^2}$$

$$\left(\begin{matrix}\text{Reference : Inequality in Triangle by Dang Ngoc Minh}-329;\\ \text{published at www.ssmrmh.ro}\end{matrix}\right)\overset{?}{\ge}2R-r$$

$$\Leftrightarrow 4R^2-7r^2\overset{?}{\ge}(2R-r)^2=4R^2-4Rr+r^2\Leftrightarrow 4Rr\overset{?}{\ge}8r^2\to\text{true via Euler}$$

$$\therefore\frac{2R-r}{r}\le\frac{max(r_a,r_b,r_c)}{r}\to(2)\ \therefore(1)\text{ and }(2)\Rightarrow$$

$$\sum_{cyc}\sqrt{\frac{1}{2}\left(\frac{n_bn_c}{r_br_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}-1\right)}\le\sqrt{\frac{max(r_a,r_b,r_c)+AI+BI+CI}{r}}\ \forall\ ABC,$$

$$''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

4387. In any ΔABC holds :

$$\frac{w_ah_a}{r_a}+\frac{w_bh_b}{r_b}+\frac{w_ch_c}{r_c}\ge p_a\cdot\sqrt{\frac{w_a}{g_a}}+p_b\cdot\sqrt{\frac{w_b}{g_b}}+p_c\cdot\sqrt{\frac{w_c}{g_c}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{w_a^2h_a^2}{r_a^2}=\frac{1}{4R^2r^2}\cdot\sum_{cyc}\left((b^2c^2+16R^2r^2-8Rrbc)\left(bc-\frac{a^2bc}{(b+c)^2}\right)\right)\to(1)$$

$$-a^2b^2c^2\sum_{cyc}\frac{bc}{(b+c)^2}=\frac{-16R^2r^2s^2\cdot\sum_{cyc}\left(bc\left(a^4+2a^2\sum_{cyc}ab+\left(\sum_{cyc}ab\right)^2\right)\right)}{4s^2(s^2+2Rr+r^2)^2}$$

$$=\frac{-4R^2r^2(s^6+(36Rr+3r^2)s^4+r^2(64R^2+16Rr+3r^2)s^2+r^3(4R+r)^3)}{(s^2+2Rr+r^2)^2}$$

$$= X_1 = -a^2b^2c^2\sum_{cyc}\frac{bc}{(b+c)^2} \to (2);\ also, -16R^2r^2.4Rrs\sum_{cyc}\frac{a}{(b+c)^2} =$$

$$-\frac{64R^3r^3}{(s^2+2Rr+r^2)^2}\left(2s^4-(3Rr+6r^2)s^2+Rr^2(4R+r)\right) = X_2 =$$

$$16R^2r^2.4Rrs\sum_{cyc}\frac{a}{(b+c)^2} \to (3);\ Also, 8Rr.16R^2r^2s^2.\sum_{cyc}\frac{1}{(b+c)^2} =$$

$$\frac{32R^3r^3}{(s^2+2Rr+r^2)^2}.(9s^4+(8Rr-6r^2)s^2+r^2(4R+r)^2) = X_3 =$$

$$8Rr.16R^2r^2s^2.\sum_{cyc}\frac{1}{(b+c)^2} \to (4)\ \&\ \because \sum_{cyc}\frac{r_a}{h_aw_a} \geq \sum_{cyc}\frac{r_a}{w_a^2} \therefore 2\sum_{cyc}\frac{w_bw_ch_bh_c}{r_br_c} \geq$$

$$\frac{2.16Rr^2s^2}{s^2+2Rr+r^2}.\frac{2r^2s^2}{R.s^2r}.\sum_{cyc}\frac{r\left(s^2+(s-a)^2+2s(s-a)\right)}{4bc(s-a)^2} \to (5)$$

$$Now\ \because (s-b)^2(s-c)^2 = a^2s^2+2abcs-2as^3+b^2c^2-2bcs^2+s^4$$

$$\therefore \frac{r}{4abc}\left(\frac{s^2}{r^4s^2}\sum_{cyc}(a(s-b)^2(s-c)^2)+2s+\frac{2}{r^2}\sum_{cyc}\left(a(s-b)(s-c)\right)\right)$$

$$= \frac{r\left(\frac{s^2\sum_{cyc}a^3+2abcs(2s)-2s^3\sum_{cyc}a^2+abc\sum_{cyc}ab-6abcs^2+2s^5}{r^4}+2s+\frac{2}{r^2}\left(-2s^3+s\sum_{cyc}a^2+12Rrs\right)\right)}{16Rrs}$$

$$\therefore (5) \Rightarrow 2\sum_{cyc}\frac{w_bw_ch_bh_c}{r_br_c} \geq \frac{8rs^2(8R^2+6Rr-r^2-s^2)}{R(s^2+2Rr+r^2)} = X_4 \to (6)\ and\ \because X_5 =$$

$$\sum_{cyc}a^3b^3 = (s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2) \therefore (1),(2),(3),(4),(5),(6) \Rightarrow$$

$$\left(\sum_{cyc}\frac{w_ah_a}{r_a}\right)^2 \geq X_1+X_2+X_3+X_4+\frac{X_5+16R^2r^2\sum_{cyc}ab-8Rr\sum_{cyc}a^2b^2}{4R^2r^2} \overset{?}{\geq}$$

$$\frac{(9R-3r)s^2-3r(4R+r)^2}{R} \Leftrightarrow s^{10}-(16Rr-5r^2)s^8-$$

$$r^2(36R^2+60Rr-10r^2)s^6+r^3(560R^3+216R^2r-52Rr^2+10r^3)s^4+$$

$$r^4(1648R^4+1696R^3r+412R^2r^2+12Rr^3+5r^4)s^2+$$

$$r^5(768R^5+1152R^4r+624R^3r^2+160R^2r^3+20Rr^4+r^5) \overset{?}{\geq} 0\ and\ \because P =$$

$$T^5+4r(16R-5r)T^4+4r^2(375R^2-255Rr+40r^2)T^3+$$

$$4r^3(3804R^3-4371R^2r+1532Rr^2-160r^3)T^2 \overset{Gerretsen}{\geq} 0\ (T=s^2-16Rr+5r^2)$$

(*via successively* implementing Gerretsen) $\therefore$ to prove $(*)$, suffices to prove :

$$\text{LHS of } (*) \overset{?}{\geq} P \Leftrightarrow (3591R^4 - 6732R^3r + 4298R^2r^2 - 1028Rr^3 + 80r^4)s^2 \overset{?}{\geq} (**)$$

$$2r(28008R^5 - 59028R^4r + 45182R^3r^2 - 16115R^2r^3 + 2690Rr^4 - 168r^5) \ \&$$

$$\text{LHS}_{(**)} \overset{\text{Rouche + Euler}}{\geq} \left(3591R^4 - 6732R^3r + 4298R^2r^2 - 1028Rr^3 + 80r^4\right)\left(16Rr - 5r^2 + \frac{r^2(R-2r)}{R-r}\right) \overset{?}{\geq}$$

$$\text{RHS}_{(**)} \Leftrightarrow 1440t^6 - 5460t^5 + 5761t^4 - 10t^3 - 2876t^2 + 1032t - 96 \overset{?}{\geq} 0$$

$$\left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(150t^5 + 1290t^4(t-2) + 601t^3 + 1192t^2 - 492t + 48) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow \left(\sum_{cyc} \frac{w_a h_a}{r_a}\right)^2 \geq \frac{(9R-3r)s^2 - 3r(4R+r)^2}{R} \rightarrow (m)$$

$$\text{Again}, \sum_{cyc} n_a \overset{\text{Bogdan Fustei}}{=} \sqrt{s^2 - \frac{rs^2}{R}.\sec^2\frac{A}{2}} \overset{\text{CBS}}{\leq} \sqrt{3\sum_{cyc}\left(s^2 - \frac{rs^2}{R}.\sec^2\frac{A}{2}\right)}$$

$$= \sqrt{\frac{(9R-3r)s^2 - 3r(4R+r)^2}{R}} \rightarrow (n) \therefore (m), (n) \Rightarrow \sum_{cyc} \frac{w_a h_a}{r_a} \geq \sum_{cyc} n_a \geq$$

$$\sum_{cyc} p_a \cdot \sqrt{\frac{w_a}{g_a}} \left(\begin{matrix}\text{Reference : Solution to Inequality in Triangle} - 136 \\ \text{by Bogdan Fustei; published at www. ssmrmh. ro}\end{matrix}\right) \forall \Delta ABC,$$

$$'' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4388. In any ΔABC holds :

$$\frac{w_a h_a}{r_a} + \frac{w_b h_b}{r_b} + \frac{w_c h_c}{r_c} \geq n_a + n_b + n_c$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{w_a^2 h_a^2}{r_a^2} = \frac{1}{4R^2r^2} \cdot \sum_{cyc}\left((b^2c^2 + 16R^2r^2 - 8Rrbc)\left(bc - \frac{a^2bc}{(b+c)^2}\right)\right) \rightarrow (1)$$

$$-a^2b^2c^2 \sum_{cyc} \frac{bc}{(b+c)^2} = \frac{-16R^2r^2s^2.\sum_{cyc}\left(bc\left(a^4 + 2a^2\sum_{cyc} ab + \left(\sum_{cyc} ab\right)^2\right)\right)}{4s^2(s^2 + 2Rr + r^2)^2}$$

$$=\frac{-4R^2r^2(s^6+(36Rr+3r^2)s^4+r^2(64R^2+16Rr+3r^2)s^2+r^3(4R+r)^3)}{(s^2+2Rr+r^2)^2}$$

$$=X_1=-a^2b^2c^2\sum_{cyc}\frac{bc}{(b+c)^2}\to(2);also,-16R^2r^2.4Rrs\sum_{cyc}\frac{a}{(b+c)^2}=$$

$$-\frac{64R^3r^3}{(s^2+2Rr+r^2)^2}\left(2s^4-(3Rr+6r^2)s^2+Rr^2(4R+r)\right)=X_2=$$

$$16R^2r^2.4Rrs\sum_{cyc}\frac{a}{(b+c)^2}\to(3);Also,8Rr.16R^2r^2s^2.\sum_{cyc}\frac{1}{(b+c)^2}=$$

$$\frac{32R^3r^3}{(s^2+2Rr+r^2)^2}.\left(2\sum_{cyc}a^2b^2-16r^2s^2+2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)+3\left(\sum_{cyc}ab\right)^2\right)$$

$$=\frac{32R^3r^3}{(s^2+2Rr+r^2)^2}.(9s^4+(8Rr-6r^2)s^2+r^2(4R+r)^2)=X_3=$$

$$8Rr.16R^2r^2s^2.\sum_{cyc}\frac{1}{(b+c)^2}\to(4)\ \&\ \because\sum_{cyc}\frac{r_a}{h_aw_a}\geq\sum_{cyc}\frac{r_a}{w_a^2}\therefore 2\sum_{cyc}\frac{w_bw_ch_bh_c}{r_br_c}\geq$$

$$\frac{2.16Rr^2s^2}{s^2+2Rr+r^2}.\frac{2r^2s^2}{R.s^2r}.\sum_{cyc}\frac{r\left(s^2+(s-a)^2+2s(s-a)\right)}{4bc(s-a)^2}\to(5)$$

$$Now\ \because(s-b)^2(s-c)^2=a^2s^2+2abcs-2as^3+b^2c^2-2bcs^2+s^4$$

$$\therefore\frac{r}{4abc}\left(\frac{s^2}{r^4s^2}\sum_{cyc}(a(s-b)^2(s-c)^2)+2s+\frac{2}{r^2}\sum_{cyc}\left(a(s-b)(s-c)\right)\right)$$

$$=\frac{r\left(\frac{s^2\sum_{cyc}a^3+2abcs(2s)-2s^3\sum_{cyc}a^2+abc\sum_{cyc}ab-6abcs^2+2s^5}{r^4}+2s+\frac{2}{r^2}\left(-2s^3+s\sum_{cyc}a^2+12Rrs\right)\right)}{16Rrs}$$

$$\therefore(5)\Rightarrow 2\sum_{cyc}\frac{w_bw_ch_bh_c}{r_br_c}\geq\frac{8rs^2(8R^2+6Rr-r^2-s^2)}{R(s^2+2Rr+r^2)}=X_4\to(6)\ and\ \because X_5=$$

$$\sum_{cyc}a^3b^3=(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\therefore(1),(2),(3),(4),(5),(6)\Rightarrow$$

$$\left(\sum_{cyc}\frac{w_ah_a}{r_a}\right)^2\geq X_1+X_2+X_3+X_4+\frac{X_5+16R^2r^2\sum_{cyc}ab-8Rr\sum_{cyc}a^2b^2}{4R^2r^2}\overset{?}{\geq}$$

$$\frac{(9R-3r)s^2-3r(4R+r)^2}{R}\Leftrightarrow s^{10}-(16Rr-5r^2)s^8-$$

$$r^2(36R^2+60Rr-10r^2)s^6+r^3(560R^3+216R^2r-52Rr^2+10r^3)s^4+$$

$$r^4(1648R^4+1696R^3r+412R^2r^2+12Rr^3+5r^4)s^2+$$

$$r^5(768R^5+1152R^4r+624R^3r^2+160R^2r^3+20Rr^4+r^5)\overset{?}{\geq}0\ and\ \because P=$$

(*) ∎

$$T^5+4r(16R-5r)T^4+4r^2(375R^2-255Rr+40r^2)T^3+$$
$$4r^3(3804R^3-4371R^2r+1532Rr^2-160r^3)T^2 \overset{\text{Gerretsen}}{\geq} 0\ (T=s^2-16Rr+5r^2)$$
$(via\ successively$ implementing Gerretsen$) \therefore$ to prove $(*)$, suffices to prove :
$$\text{LHS of } (*) \overset{?}{\geq} P \Leftrightarrow (3591R^4-6732R^3r+4298R^2r^2-1028Rr^3+80r^4)s^2 \overset{?}{\geq} (**)$$
$$2r\left(28008R^5-59028R^4r+45182R^3r^2-16115R^2r^3+2690Rr^4-168r^5\right)\ \&$$
$$\text{LHS}_{(**)} \overset{\text{Rouche + Euler}}{\geq} \left(3591R^4-6732R^3r+4298R^2r^2-1028Rr^3+80r^4\right)\left(16Rr-5r^2+\frac{r^2(R-2r)}{R-r}\right) \overset{?}{\geq}$$
$$\text{RHS}_{(**)} \Leftrightarrow 1440t^6-5460t^5+5761t^4-10t^3-2876t^2+1032t-96 \overset{?}{\geq} 0$$
$$\left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)\left(150t^5+1290t^4(t-2)+601t^3+1192t^2-492t+48\right) \overset{?}{\geq} 0$$
$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow \left(\sum_{cyc}\frac{w_a h_a}{r_a}\right)^2 \geq \frac{(9R-3r)s^2-3r(4R+r)^2}{R} \rightarrow (m)$$
$$\text{Again, } \sum_{cyc} n_a \overset{\text{Bogdan Fustei}}{=} \sqrt{s^2-\frac{rs^2}{R}.\sec^2\frac{A}{2}} \overset{CBS}{\leq} \sqrt{3\sum_{cyc}\left(s^2-\frac{rs^2}{R}.\sec^2\frac{A}{2}\right)}$$
$$= \sqrt{\frac{(9R-3r)s^2-3r(4R+r)^2}{R}} \rightarrow (n) \therefore (m)\ and\ (n) \Rightarrow \text{QED}, '' = '' \text{ iff } a=b=c$$

4389. *In any ΔABC the following relationship holds:*

$$\frac{1}{h_a m_a}+\frac{1}{h_b m_b}+\frac{1}{h_c m_c} \leq \frac{1}{h_a h_b}+\frac{1}{h_b h_c}+\frac{1}{h_c h_a}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Jenish Rijal-Bardiya-Nepal

Lemma: $m_a \geq \sqrt{s(s-a)}$ $(and\ analogs)$

Proof: $m_a^2 = \frac{2b^2+2c^2-a^2}{4} = \frac{(b+c)^2-a^2+(b-c)^2}{4} \overset{(b-c)^2\geq 0}{\geq} \frac{(b+c)^2-a^2}{4}$
$= s(s-a)$

Here, $\sum_{cyc}\frac{1}{h_a m_a} = \sum_{cyc}\frac{a}{2\Delta m_a}$ *and* $\sum_{cyc}\frac{1}{h_a h_b} = \sum_{cyc}\frac{a\cdot b}{2\Delta\cdot 2\Delta} = \frac{ab+bc+ca}{4\Delta^2}$

$$\textit{Also, } \sum_{cyc}\frac{1}{h_a m_a}=\sum_{cyc}\frac{a}{2\Delta m_a}\overset{Lemma}{\geq}\sum_{cyc}\frac{a}{2\Delta\sqrt{s(s-a)}}=\sum_{cyc}\frac{2a\cdot\sqrt{(s-b)(s-c)}}{4\Delta^2}$$

$$\textit{Via } \sum_{cyc}a^2=2(s^2-r^2-4Rr) \textit{ and } \sum_{cyc}(s-b)(s-c)=r^2+4Rr, \textit{we have:}$$

$$\sum_{cyc}\frac{2a\cdot\sqrt{(s-b)(s-c)}}{4\Delta^2}\overset{CBS}{\leq}\frac{2\sqrt{[\sum a^2]\cdot[\sum(s-b)(s-c)]}}{4\Delta^2}$$

$$=\frac{2\sqrt{[2(s^2-r^2-4Rr)]\cdot[r^2+4Rr]}}{4\Delta^2}=$$

$$=\frac{2\sqrt{(s^2-r^2-4Rr)\cdot 2(r^2+4Rr)}}{4\Delta^2}\overset{AM-GM}{\leq}\frac{s^2-r^2-4Rr+2(r^2+4Rr)}{4\Delta^2}\overset{\sum ab=s^2+r^2+4Rr}{=}\frac{ab+bc+ca}{4\Delta^2}$$

$$\textit{Therefore, } \frac{1}{h_a m_a}+\frac{1}{h_b m_b}+\frac{1}{h_c m_c}\leq\frac{1}{h_a h_b}+\frac{1}{h_b h_c}+\frac{1}{h_c h_a}$$

Equality holds if and only if the triangle is equilateral.

4390. In any ΔABC the following relationship holds :

$$\frac{s^2-2Rr-(23-12\sqrt{3})r^2}{2R}\leq s\leq 2R+r+\frac{9(3\sqrt{3}-5)r^2}{7R-5r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$s\overset{?}{\leq}2R+r+\frac{9(3\sqrt{3}-5)r^2}{7R-5r}\Leftrightarrow s\overset{?}{\leq}\frac{14R^2-3Rr-50r^2}{7R-5r}+\frac{27\sqrt{3}.r^2}{7R-5r}\overset{\text{squaring}}{\Leftrightarrow}$$

$$s^2\overset{?}{\leq}\frac{(14R^2-3Rr-50r^2)^2}{(7R-5r)^2}+\frac{2187r^4}{(7R-5r)^2}+\frac{54\sqrt{3}.r^2(14R^2-3Rr-50r^2)}{7R-5r} \text{ and } \because$$

$$54\sqrt{3}>\frac{187}{2} \text{ and } \because 14R^2-3Rr-50r^2=(R-2r)(14R+25r)\overset{\text{Euler}}{\geq}0$$

$$\therefore \text{it suffices to prove : } s^2\overset{?}{\underset{(*)}{\leq}}\frac{(14R^2-3Rr-50r^2)^2}{(7R-5r)^2}+\frac{2187r^4}{(7R-5r)^2}+$$

$$\frac{187r^2(14R^2-3Rr-50r^2)}{2(7R-5r)} \text{ and now, } s^2\overset{\text{Rouche}}{\leq}$$

$$2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\overset{?}{\leq}\text{RHS of }(*)$$

$$\Leftrightarrow 196R^4-868R^3r+1234R^2r^2-601Rr^3+74r^4\overset{?}{\geq}$$

$$4(R-2r).\sqrt{R^2-2Rr}.(7R-5r)^2$$

$$\Leftrightarrow (R-2r)(196R^3-476R^2r+282Rr^2-37r^3)\overset{?}{\geq}0\ (**)$$

$4(R-2r).\sqrt{R^2-2Rr}.(7R-5r)^2$ *and* $\because 196R^3-476R^2r+282Rr^2-37r^3=$

$$(R-2r)(196R^2-84Rr+114r^2)+191r^3\overset{\text{Euler}}{\geq}191r^3>0$$

$\therefore$ **in order to prove** $(**)$, **it suffices to prove :**

$$(196R^3-476R^2r+282Rr^2-37r^3)^2\overset{?}{>}16(R^2-2Rr)(7R-5r)^4$$

$$\Leftrightarrow 8232t^3-7252t^2-868t+1369\overset{?}{>}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(8232t^2+9212t+17556)+36481\overset{?}{>}0\rightarrow \text{true}\ \because t\overset{\text{Euler}}{\geq}2$$

$\Rightarrow (**)\Rightarrow (*)$ **is true** $\therefore \boxed{s\leq 2R+r+\frac{9(3\sqrt{3}-5)r^2}{7R-5r}}$ *and now,*

$$\frac{s^2-2Rr-(23-12\sqrt{3})r^2}{2R}\overset{?}{\leq}s\Leftrightarrow s^2-2Rr-23r^2+12\sqrt{3}.r^2\overset{?}{\leq}2Rs$$

$$\Leftrightarrow (s^2-2Rr-23r^2)^2+432r^4+24\sqrt{3}.r^2(s^2-2Rr-23r^2)\overset{?}{\leq}4R^2s^2\ \textit{and}\ \because$$

$$24\sqrt{3}<\frac{125}{3}\ \textit{and}\ s^2-2Rr-23r^2\overset{\text{Gerretsen}}{\geq}14Rr-28r^2=14r(R-2r)\overset{\text{Euler}}{\geq}0$$

$\therefore$ **it suffices to prove :**

$$(s^2-2Rr-23r^2)^2+432r^4+\frac{125}{3}.r^2(s^2-2Rr-23r^2)\overset{?}{\leq}4R^2s^2$$

$$\Leftrightarrow -3s^4+(12R^2+12Rr+13r^2)s^2-r^2(12R^2+26Rr+8r^2)\overset{?}{\geq}0\ (***)$$

Now, Rouche $\Rightarrow s^2-(m-n)\geq 0$ *and* $s^2-(m+n)\leq 0$, **where** $m=$

$2R^2+10Rr-r^2$ & $n=2(R-2r)\sqrt{R^2-2Rr}\ \therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$

$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\leq 0$

$\Rightarrow P=-3(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\geq 0$

$\therefore$ **in order to prove** $(***)$, **it suffices to prove : LHS of** $(***)\overset{?}{\geq}P$

$$\Leftrightarrow 192R^3+132R^2r+10Rr^2-5r^3\overset{?}{\geq}(48R-19r)s^2\ \textit{and finally,}\ (****)$$

$$(48R-19r)s^2\overset{\text{Gerretsen}}{\leq}(48R-19r)(4R^2+4Rr+3r^2)\overset{?}{\leq}\text{LHS of}\ (****)$$

$$\Leftrightarrow 2r(R-2r)(8R-13r)\overset{?}{\geq}0\rightarrow\text{true}\ \because R\overset{\text{Euler}}{\geq}2r\Rightarrow (****)\Rightarrow (***)\ \textbf{is true}$$

$$\therefore \boxed{s\geq\frac{s^2-2Rr-(23-12\sqrt{3})r^2}{2R}}\ \textit{and so,}$$

$$\frac{s^2-2Rr-(23-12\sqrt{3})r^2}{2R}\leq s\leq 2R+r+\frac{9(3\sqrt{3}-5)r^2}{7R-5r}\ \forall\ \Delta ABC,$$

$''=''$ **iff** ΔABC **is equi***lateral* **(QED)**

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4391. In any ΔABC the following relationship holds :

$$h_a + h_b + h_c \le s - \frac{27Rr}{4s} + \left(\frac{18-3\sqrt{3}}{2}\right)r$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} h_a \overset{?}{\le} s - \frac{27Rr}{4s} + \left(\frac{18-3\sqrt{3}}{2}\right)r$$

$$\Leftrightarrow \frac{s^2+4Rr+r^2}{2R} - 9r + \frac{3\sqrt{3}}{2}r \overset{?}{\le} \frac{4s^2-27Rr}{4s}$$

$$\Leftrightarrow \frac{s^2-14Rr+r^2}{2R} + \frac{3\sqrt{3}}{2}r \overset{?}{\le} \frac{4s^2-27Rr}{4s}$$

$$\Leftrightarrow \frac{(s^2-14Rr+r^2)^2}{4R^2} + \frac{27r^2}{4} + \frac{3\sqrt{3}.r(s^2-14Rr+r^2)}{2R} \overset{?}{\le} \frac{(4s^2-27Rr)^2}{16s^2}$$

$$\text{and } \because \frac{3\sqrt{3}}{2} < \frac{13}{5} \text{ and } \because s^2-14Rr+r^2 \overset{\text{Gerretsen}}{\ge} 2r(R-2r) \overset{\text{Euler}}{\ge} 0$$

$$\therefore \text{it suffices to prove :}$$

$$\frac{(s^2-14Rr+r^2)^2}{4R^2} + \frac{27r^2}{4} + \frac{13r(s^2-14Rr+r^2)}{5R} \overset{?}{\le} \frac{(4s^2-27Rr)^2}{16s^2}$$

$$\Leftrightarrow -20s^6 + (80R^2+352Rr-40r^2)s^4 -$$

$$r(1080R^3+1548R^2r-352Rr^2+20r^3)s^2 + 3645R^4r^2 \overset{?}{\underset{①}{\ge}} 0$$

Now, Rouche $\Rightarrow s^2-(m-n) \ge 0$ and $s^2-(m+n) \le 0$, where $m = 2R^2+10Rr-r^2$ & $n = 2(R-2r)\sqrt{R^2-2Rr} \therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right) \le 0$

$$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \le 0 \Rightarrow s^4 - (4R^2+20Rr-2r^2)s^2 + r(4R+r)^3 \overset{(*)}{\le} 0$$

$$\Rightarrow P = -20s^2(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3) \ge 0$$

$$\therefore \text{in order to prove } ①, \text{it suffices to prove : LHS of } ① \overset{?}{\ge} P$$

$$\Leftrightarrow -48s^4 + (200R^2-588Rr+592r^2)s^2 + 3645R^3r \overset{?}{\underset{②}{\ge}} 0 \text{ and again,}$$

$$Q = -48(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3) \overset{\text{via } (*)}{\ge} 0$$

$$\therefore \text{in order to prove } ②, \text{it suffices to prove : LHS of } ② \overset{?}{\ge} Q$$

$$\Leftrightarrow (8R^2-1548Rr+688r^2)s^2 + r(6717R^3+2304R^2r+576Rr^2+48r^3) \overset{?}{\underset{③}{\ge}} 0$$

and it's trivially true if : $8R^2-1548Rr+688r^2 \ge 0$ and when :

$$8R^2-1548Rr+688r^2<0,\text{then : LHS of } (3) \overset{\text{Gerretsen}}{\geq}$$
$$(8R^2-1548Rr+688r^2)(4R^2+4Rr+3r^2)+$$
$$r(6717R^3+2304R^2r+576Rr^2+48r^3)\overset{?}{\geq}0$$
$$\Leftrightarrow 32t^4+557t^3-1112t^2-1316t+2112\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)(32t^3+621(t^2-4)+130t+1428)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$
$$\Rightarrow (3)\Rightarrow(2)\Rightarrow(1)\text{ is true and so,}$$
$$h_a+h_b+h_c\leq s-\frac{27Rr}{4s}+\left(\frac{18-3\sqrt{3}}{2}\right)r\ \forall\ \Delta ABC,$$
$$''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4392. In any ΔABC holds :

$$\frac{\sqrt{s^2-9Rr}}{r}\leq\frac{m_a}{r_a}+\frac{m_b}{r_b}+\frac{m_c}{r_c}\leq\frac{\sqrt{s^2-7Rr-4r^2}}{r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{m_a^2}{r_a^2}=\sum_{cyc}\left(\frac{2\sum_{cyc}a^2-3a^2}{4r^2s^2}.(s-a)^2\right)=$$
$$\frac{4(s^2-4Rr-r^2)(s^2-8Rr-2r^2)-3\left(s^2\sum_{cyc}a^2-2s\sum_{cyc}a^3+2\sum_{cyc}a^2b^2-16r^2s^2\right)}{4r^2s^2}$$
$$=\frac{4(s^2-4Rr-r^2)(s^2-8Rr-2r^2)}{4r^2s^2}-$$
$$\frac{3(2s^2(s^2-4Rr-r^2)-4s^2(s^2-6Rr-3r^2)+2(s^2+4Rr+r^2)^2-32Rrs^2-16r^2s^2)}{4r^2s^2}\overset{(1)}{=}$$
$$\sum_{cyc}\frac{m_a^2}{r_a^2};\text{Now, }2m_bm_c\overset{?}{\geq}\frac{9}{4}a^2+h_a^2-m_b^2-m_c^2$$
$$=\frac{a^2(5a^2-b^2-c^2)+2\sum_{cyc}a^2b^2-\sum_{cyc}a^4}{4a^2}=\frac{a^2(4a^2+b^2+c^2)-(b^2-c^2)^2}{4a^2}$$
$$\text{and it's trivially true if : } a^2(4a^2+b^2+c^2)-(b^2-c^2)^2\leq 0\text{ and when :}$$
$$a^2(4a^2+b^2+c^2)-(b^2-c^2)^2>0,\text{then we are to prove :}$$
$$4a^4(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)\overset{?}{\geq}(a^2(4a^2+b^2+c^2)-(b^2-c^2)^2)^2\Leftrightarrow$$
$$(b^2-c^2)^2(-a^4+2a^2(b^2+c^2)-b^4-c^4+2b^2c^2)\overset{?}{\geq}0\Leftrightarrow(b^2-c^2)^2.16F^2\overset{?}{\geq}0$$
$$\to\text{true}\therefore 2m_bm_c\geq\frac{9}{4}a^2+h_a^2-m_b^2-m_c^2=\frac{6a^2-\sum_{cyc}a^2}{4}+h_a^2\text{ and analogs}$$

$$\Rightarrow \sum_{cyc}\frac{2m_bm_c}{r_br_c}\geq \sum_{cyc}\left(\frac{6a^2-\sum_{cyc}a^2}{4}+h_a^2\right)\frac{(-s^2+sa+bc)}{r^2s^2}$$

$$=\frac{6R^2\left(-s^2\sum_{cyc}a^2+s\sum_{cyc}a^3+4Rrs.\,2s-2(s^2-4Rr-r^2)(4Rr+r^2)\right)-s^2\sum_{cyc}a^2b^2+4Rrs^2(s^2+4Rr+r^2)+\sum_{cyc}a^3b^3}{4R^2r^2s^2}$$

$$=\frac{6}{4r^2s^2}\left(\begin{matrix}-2s^2(s^2-4Rr-r^2)+2s^2(s^2-6Rr-3r^2)+8Rrs^2-\\2(s^2-4Rr-r^2)(4Rr+r^2)\end{matrix}\right)+$$

$$\frac{1}{4R^2r^2s^2}\left(\begin{matrix}-s^2((s^2+4Rr+r^2)^2-16Rrs^2)+4Rrs^2(s^2+4Rr+r^2)+\\(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\end{matrix}\right)\overset{②}{\geq}\sum_{cyc}\frac{2m_bm_c}{r_br_c}$$

$$\therefore ①+②\overset{\text{upon simplification}}{\Rightarrow}\left(\sum_{cyc}\frac{m_a}{r_a}\right)^2=\sum_{cyc}\frac{m_a^2}{r_a^2}+\sum_{cyc}\frac{2m_bm_c}{r_br_c}\geq$$

$$\frac{(4R^2+r^2)s^4-r(32R^3+32R^2r+4Rr^2-2r^3)s^2+r^2(64R^4+96R^3r+52R^2r^2+12Rr^3+r^4)}{4R^2r^2s^2}$$

$$\overset{?}{\geq}\frac{s^2-9Rr}{r^2}\Leftrightarrow rs^4+(4R^3-32R^2r-4Rr^2+2r^3)s^2+$$

$$r(64R^4+96R^3r+52R^2r^2+12Rr^3+r^4)\overset{?}{\underset{(*)}{\geq}}0\ \&\ \because (s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0$$

$$\therefore \text{in order to prove } (*), \text{it suffices to prove : LHS of } (*)\overset{?}{\geq} r(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow \overbrace{(R^3-8R^2r+7Rr^2-2r^3)}^{Y}s^2+r\overbrace{(16R^4+24R^3r-51R^2r^2+43Rr^3-6r^4)}^{X>0}$$

$$\overset{?}{\underset{(**)}{\geq}}0;\ \text{trivially true if : } Y\geq 0\ \&\ \text{when : } Y<0,\text{then : LHS of } (**)\overset{\text{Gerretsen}}{\geq}$$

$$(R^3-8R^2r+7Rr^2-2r^3)(4R^2+4Rr+3r^2)+X\overset{?}{\geq}0$$

$$\Leftrightarrow (t-2)(2t^4+2t^3(t-2)+15t(t-2)+5t+6)\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\rightarrow \text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\Rightarrow (**)\Rightarrow (*)\text{ is true }\therefore \frac{m_a}{r_a}+\frac{m_b}{r_b}+\frac{m_c}{r_c}\geq\frac{\sqrt{s^2-9Rr}}{r}\ \textit{and again},\sum_{cyc}\frac{2m_bm_c}{r_br_c}\leq$$

$$\sum_{cyc}\frac{2(2a^2+bc)(-s^2+sa+bc)}{4r^2s^2}=$$

$$\frac{-s^2\left(4s^2+6(s^2-4Rr-r^2)\right)+8s^2(s^2-6Rr-3r^2)+24Rrs^2+32Rrs^2+2(s^2+4Rr+r^2)^2-32Rrs^2}{4r^2s^2}\rightarrow ③$$

$$\therefore (1)+(3)\Rightarrow\left(\sum_{cyc}\frac{m_a}{r_a}\right)^2\le\frac{s^4-(8Rr+5r^2)s^2+r^2(4R+r)^2}{r^2s^2}\overset{?}{\le}\frac{s^2-7Rr-4r^2}{r^2}$$

$$\Leftrightarrow (R+r)s^2\overset{?}{\ge}r(4R+r)^2\to true\because (R+r)s^2-r(4R+r)^2\overset{Gerretsen}{\ge}$$

$$(R+r)(16Rr-5r^2)-r(4R+r)^2=12r^2(R-2r)\overset{Euler}{\ge}0$$

$$\therefore\frac{m_a}{r_a}+\frac{m_b}{r_b}+\frac{m_c}{r_c}\le\frac{\sqrt{s^2-7Rr-4r^2}}{r};\ QED,\ ''=''\ iff\ \Delta ABC\ is\ equilateral$$

4393. *In any ΔABC the following relationship holds:*

$$\frac{a^2sinA}{a+sinA}+\frac{b^2sinB}{b+sinB}+\frac{c^2sinC}{c+sinC}\le\frac{9R^3}{2r(4r+1)}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{a^2sinA}{a+sinA}+\frac{b^2sinB}{b+sinB}+\frac{c^2sinC}{c+sinC}=\sum_{cyc}\frac{a^2sinA}{a+sinA}=$$

$$=\sum_{cyc}\frac{4R^2\sin^3A}{2RsinA+sinA}=\frac{4R^2}{2R+1}\sum_{cyc}\sin^2A=$$

$$=\frac{4R^2}{2R+1}\times\frac{p^2-r^2-4Rr}{2R^2}\overset{Gerretsen}{\le}\frac{4R^2}{2R+1}\times\frac{4R^2+4Rr+3r^2-r^2-4Rr}{2R^2}$$

$$=\frac{4(2R^2+r^2)}{2R+1}\overset{Euler}{\le}\frac{9R^2}{2R+1}=\frac{9R^3}{R(2R+1)}\overset{Euler}{\le}\frac{9R^3}{2r(4r+1)}$$

Equality holds for $a=b=c$.

4394. In any acute ΔABC the following relationship holds:

$$\cos^2A\tan A+\cos^2B\tan B+\cos^2C\tan C\le\frac{3\sqrt{3}}{4}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

Let be $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R};f(x)=\cos^2x\tan x$

$$f(x)=\cos^2x\cdot\frac{\sin x}{\cos x}=\sin x\cos x=\frac{1}{2}\sin 2x$$

$$f'(x)=\frac{1}{2}\cdot 2\cos 2x=\cos 2x,\qquad f''(x)=-2\sin 2x<0$$

f concave. By Jensen's inequality:

$$f(A)+f(B)+f(C)\le 3f\left(\frac{A+B+C}{3}\right)$$

$$\cos^2 A\tan A+\cos^2 B\tan B+\cos^2 C\tan C\le$$

$$\le 3\cos^2\left(\frac{A+B+C}{3}\right)\tan\left(\frac{A+B+C}{3}\right)=3\cos^2\frac{\pi}{3}\cdot\tan\left(\frac{\pi}{3}\right)=3\cdot\left(\frac{1}{2}\right)^2\cdot\sqrt{3}=\frac{3\sqrt{3}}{4}$$

Equality holds for $A=B=C=\frac{\pi}{3}$.

4395. In ΔABC te following relationship holds:

$$\sum_{cyc}\frac{r\cdot r_a}{r_b\cdot r_c}\le\frac{3R}{2r}-2$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$r\cdot r_a=\frac{F}{p}\cdot\frac{F}{p-a}=\frac{F^2}{p(p-a)};\qquad r_b\cdot r_c=p(p-a)$$

$$\sum_{cyc}\frac{r\cdot r_a}{r_b\cdot r_c}=\sum_{cyc}\frac{F^2}{p(p-a)}\cdot\frac{1}{p(p-a)}=\sum_{cyc}\frac{F^2}{p^2(p-a)^2}=$$

$$=F^2\cdot\frac{(p-a)^2(p-b)^2+(p-b)^2(p-c)^2+(p-c)^2(p-a)^2}{p^2(p-a)^2(p-b)^2(p-c)^2}=$$

$$=F^2\cdot\frac{r^2(4R+r)^2-2p^2r^2}{F^4}=$$

$$=\frac{r^2(4R+r)^2-2p^2r^2}{p^2r^2}=\frac{(4R+r)^2}{p^2}-2\ \le^{Euler}\ \frac{81R^2}{4p^2}-2\le$$

$$\le^{(2p^2\ge 27Rr)}\ \frac{81R^2}{2\cdot 27Rr}-2=\frac{3R}{2r}-2$$

$$Equality\ holds\ for\ a=b=c.$$

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4396. In any ΔABC the following relationship holds :

$$\sum_{cyc}\left(\cos A\sqrt{\sin\frac{A}{2}}\right)\leq\frac{3\sqrt{2}}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(\cos A.\sqrt{\sin\frac{A}{2}}\right)=\sum_{cyc}\left(\left(1-2\sin^2\frac{A}{2}\right).\sqrt{\sin\frac{A}{2}}\right)$$

$$=\sum_{cyc}\sqrt{\sin\frac{A}{2}}-2\sum_{cyc}\left(\sin^2\frac{A}{2}.\sqrt{\sin\frac{A}{2}}\right)\overset{\text{CBS + Jensen}}{\leq}\sqrt{3.\frac{3}{2}}-2\sum_{cyc}\frac{\sin^4\frac{A}{2}}{\sin\frac{A}{2}.\sqrt{\sin\frac{A}{2}}}$$

$$\overset{\text{Bergstrom}}{\leq}\frac{3.\sqrt{2}}{2}-\frac{2\left(\sum_{cyc}\sin^2\frac{A}{2}\right)^2}{\sum_{cyc}\left(\sin\frac{A}{2}.\sqrt{\sin\frac{A}{2}}\right)}\overset{\text{CBS}}{\leq}\frac{3.\sqrt{2}}{2}-\frac{2\left(\sum_{cyc}\sin^2\frac{A}{2}\right)^2}{\sqrt{\sum_{cyc}\sin^2\frac{A}{2}}.\sqrt{\sum_{cyc}\sin\frac{A}{2}}}$$

$$\overset{\text{Jensen}}{\leq}\frac{3.\sqrt{2}}{2}-\frac{2\left(\frac{2R-r}{2R}\right)^{\frac{3}{2}}}{\sqrt{\frac{3}{2}}}\leq\frac{3.\sqrt{2}}{2}-\frac{2.\left(\frac{3}{4}\right)^{\frac{3}{2}}}{\sqrt{\frac{3}{2}}}\left(\because\frac{2R-r}{2R}\overset{\text{Euler}}{\geq}\frac{3}{4}\right)=\frac{3.\sqrt{2}}{2}-\frac{2.\left(\frac{3}{4}\right).\sqrt{3}}{2.\sqrt{\frac{3}{2}}}$$

$$=\frac{3.\sqrt{2}}{2}-\frac{3.\sqrt{2}}{4}=\frac{3.\sqrt{2}}{4}\therefore\sum_{cyc}\left(\cos A.\sqrt{\sin\frac{A}{2}}\right)\leq\frac{3.\sqrt{2}}{4}\ \forall\Delta ABC,$$

$''=''$ iff ΔABC is equilateral (QED)

4397. In ΔABC the following relationship holds:

$$\cos\frac{A}{2}\sqrt{\cos\frac{A}{2}}+\cos\frac{B}{2}\sqrt{\cos\frac{B}{2}}+\cos\frac{C}{2}\sqrt{\cos\frac{C}{2}}\leq 3\left(\frac{\sqrt{3}}{2}\right)^{\frac{3}{2}}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

Let be $f:(0,\pi)\to\mathbb{R};f(x)=\left(\cos\frac{x}{2}\right)^{\frac{3}{2}},f'(x)=-\frac{3}{4}\sin\frac{x}{2}\left(\cos\frac{x}{2}\right)^{\frac{1}{2}},$

$$f''(x) = -\left(\frac{3}{8}\right)\left(\cos\frac{x}{2}\right)^{\frac{3}{2}} < 0$$

f concave. By Jensen's inequality:

$$f(A) + f(B) + f(C) \leq 3f\left(\frac{A+B+C}{3}\right)$$

$$\cos\frac{A}{2}\sqrt{\cos\frac{A}{2}} + \cos\frac{B}{2}\sqrt{\cos\frac{B}{2}} + \cos\frac{C}{2}\sqrt{\cos\frac{C}{2}} \leq 3\left[\cos\left(\frac{A+B+C}{6}\right)\right]^{\frac{3}{2}} =$$

$$= 3\left(\cos\frac{\pi}{6}\right)^{\frac{3}{2}} = 3\cdot\left(\frac{\sqrt{3}}{2}\right)^{\frac{3}{2}}$$

Equality holds for $A = B = C = \frac{\pi}{3}$.

4398. In ΔABC the following relationship holds:

$$\sin A\sqrt{\sin A} + \sin B\sqrt{\sin B} + \sin C\sqrt{\sin C} \leq 3\cdot\left(\frac{\sqrt{3}}{2}\right)^{\frac{3}{2}}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

Let be $f:(0,\pi)\to\mathbb{R}; f(x) = (\sin x)^{\frac{3}{2}},\ f'(x) = \frac{3}{2}\cos x\cdot(\sin x)^{\frac{1}{2}}$

$$f''(x) = -\frac{3}{2}\left(\sin x\sqrt{\sin x} + \frac{1}{2}\cos x\sqrt{\sin x}\right) \leq 0$$

f concave. By Jensen's inequality:

$$f(A) + f(B) + f(C) \leq 3f\left(\frac{A+B+C}{3}\right)$$

$$(\sin A)^{\frac{3}{2}} + (\sin B)^{\frac{3}{2}} + (\sin C)^{\frac{3}{2}} \leq 3\left(\sin\frac{\pi}{3}\right)^{\frac{3}{2}}$$

$$\sin A\sqrt{\sin A} + \sin B\sqrt{\sin B} + \sin C\sqrt{\sin C} \leq 3\cdot\left(\frac{\sqrt{3}}{2}\right)^{\frac{3}{2}}$$

Equality holds for $A = B = C = \frac{\pi}{3}$.

4399. In acute ΔABC the following relationship holds:

$$\tan^{2007} A + \tan^{2007} B + \tan^{2007} C \geq 3 \cdot \left(\sqrt{3}\right)^{2007}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

Let be $f: \left(0, \frac{\pi}{2}\right) \to \mathbb{R}; f(x) = \tan^{2007} x,\ f'(x) = 2007 \cdot \frac{1}{\cos^2 x} \cdot \tan^{2006} x$

$$f''(x) = 2007 \cdot \frac{\sin 2x}{\cos^4 x} \tan^{2006} x + 2007 \cdot 2006 \cdot \frac{1}{\cos^4 x} \cdot \tan^{2005} x$$

$f''(x) > 0 \Rightarrow f$ convex. By Jensen's inequality:

$$\tan^{2007} A + \tan^{2007} B + \tan^{2007} C =$$

$$= f(A) + f(B) + f(C) \overset{JENSEN}{\geq} 3f\left(\frac{A+B+C}{3}\right) =$$

$$= 3f\left(\frac{\pi}{3}\right) = 3\tan^{2007}\left(\frac{\pi}{3}\right) = 3 \cdot \left(\sqrt{3}\right)^{2007}$$

Equality holds for $A = B = C = \frac{\pi}{3}$.

4400. In any acute Δ ABC the following relationship holds :

$$s.\sqrt{\frac{2R}{r} - 1} \geq \frac{w_a r_a}{h_a} + \frac{w_b r_b}{h_b} + \frac{w_c r_c}{h_c} \geq \frac{9}{2} R$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \left(r_a \cos^2 \frac{B-C}{2}\right) = \frac{1}{abc} \cdot \sum_{cyc} \left(\frac{s^2 + (s-a)^2 + 2s(s-a)}{a} (s-b)(s-c) r_a\right) \to (1)$$

$$\bullet\ \frac{s^2}{4Rrs} \cdot \sum_{cyc} \left((-s^2 + sa + bc)\frac{r_a}{a}\right) = \frac{s}{4Rr}\left(-s^2 \cdot \frac{s}{4R} \cdot \frac{s^2 + (4R+r)^2}{s^2} + s(4R+r)\right) +$$

$$\frac{s}{4Rr} \cdot \frac{rs}{4Rrs.r^2 s} \cdot \begin{pmatrix} -s^2((s^2+4Rr+r^2)^2 - 16Rrs^2) + 4Rrs^2(s^2+4Rr+r^2) + \\ (s^2+4Rr+r^2)^3 - 24Rrs^2(s^2+2Rr+r^2) \end{pmatrix}$$

$$= \frac{(4R+r)^3 - (8R-r)s^2}{16R^2}$$

$$\therefore \frac{1}{abc} \cdot \sum_{cyc} \frac{s^2(s-b)(s-c)r_a}{a} = \frac{(4R+r)^3 - (8R-r)s^2}{16R^2} \to (2)$$

$$\bullet\ \frac{1}{abc}\cdot\sum_{cyc}\frac{(s-a)^2(s-b)(s-c)r_a}{a}=\frac{r^2s}{4Rrs.\,4Rrs}\cdot\sum_{cyc}\left((s-a)\frac{rs}{s-a}.bc\right)$$

$$\therefore\frac{1}{abc}\cdot\sum_{cyc}\frac{(s-a)^2(s-b)(s-c)r_a}{a}=\frac{r(s^2+4Rr+r^2)}{16R^2}\to(3)$$

$$\bullet\ \frac{1}{abc}\cdot\sum_{cyc}\frac{2s(s-a)(s-b)(s-c)r_a}{a}=\frac{2r(s^2+(4R+r)^2)}{16R^2}\to(4)$$

$$\therefore(1)\text{ combined with }(2)+(3)+(4)\Rightarrow\sum_{cyc}\left(r_a\cos^2\frac{B-C}{2}\right)=$$

$$\frac{16R^3+20R^2r+8Rr^2+r^3-(2R-r)s^2}{4R^2}\to(5)\ \textit{and now},\ \sum_{cyc}\frac{w_ar_a}{h_a}=\sum_{cyc}\frac{r_a}{\cos\frac{B-C}{2}}=$$

$$\sum_{cyc}\frac{r_a^2}{r_a\cos\frac{B-C}{2}}\overset{\text{Bergstrom}}{\geq}\frac{(4R+r)^2}{\sum_{cyc}\left(r_a\cos\frac{B-C}{2}\right)}\overset{\text{CBS}}{\geq}\frac{(4R+r)^2}{\sqrt{4R+r}.\sqrt{\sum_{cyc}\left(r_a\cos^2\frac{B-C}{2}\right)}}\overset{\text{via }(5)}{=}$$

$$\frac{(4R+r)^2}{\sqrt{4R+r}.\sqrt{\frac{16R^3+20R^2r+8Rr^2+r^3-(2R-r)s^2}{4R^2}}}\overset{\text{Walker}}{\geq}$$

$$\frac{(4R+r)^2}{\sqrt{4R+r}.\sqrt{\frac{16R^3+20R^2r+8Rr^2+r^3-(2R-r)(2R^2+8Rr+3r^2)}{4R^2}}}$$

$$\overset{?}{\geq}\frac{9}{2}R\Leftrightarrow 26t^3+141t^2-309t-154\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(26t^2+193t+77)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\Rightarrow\boxed{\frac{w_ar_a}{h_a}+\frac{w_br_b}{h_b}+\frac{w_cr_c}{h_c}\geq\frac{9}{2}R\ \forall\ acute\ \Delta\ ABC}\ \textit{and again},\ \sum_{cyc}\frac{w_ar_a}{h_a}=\sum_{cyc}\frac{r_a}{\cos\frac{B-C}{2}}$$

$$\overset{\text{CBS}}{\leq}\sqrt{\sum_{cyc}\frac{r_a}{\cos^2\frac{B-C}{2}}}.\sqrt{4R+r}=\sqrt{\sum_{cyc}\frac{r_a.a^2bc}{(b+c)^2(s-b)(s-c)}}.\sqrt{4R+r}\overset{\text{AM-GM}}{\leq}$$

$$\sqrt{\sum_{cyc}\frac{\frac{rs}{s-a}.a^2(s-a)}{4(s-b)(s-c)(s-a)}}.\sqrt{4R+r}=\sqrt{\frac{(s^2-4Rr-r^2)(4R+r)}{2r}}\overset{?}{\leq}s.\sqrt{\frac{2R}{r}-1}$$

$$\Leftrightarrow(4R+r)^2\overset{?}{\geq}3s^2\to\text{true via Doucet (or Trucht)}$$

$$\therefore\boxed{\frac{w_ar_a}{h_a}+\frac{w_br_b}{h_b}+\frac{w_cr_c}{h_c}\leq s.\sqrt{\frac{2R}{r}-1}\ \forall\ \Delta\ ABC}\ \textit{and so},$$

$$\text{s.}\sqrt{\frac{2R}{r}-1}\geq\frac{w_a r_a}{h_a}+\frac{w_b r_b}{h_b}+\frac{w_c r_c}{h_c}\geq\frac{9}{2}R\ \forall\ acute\ \Delta ABC,$$
$$''=''\ \text{iff}\ \Delta ABC\ \text{is equi}\textit{lateral}\ (QED)$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru