

RMM - Cyclic Inequalities Marathon 2201 - 2300

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2201. If $a,b,c>0$ then prove that :

$$\sum_{cyc}\frac{a(b+c)}{a^2+bc}\geq 1+\frac{16abc}{(a+b)(b+c)(c+a)}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Assigning } b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0,$$
$$y+z-x=2a>0 \text{ and } z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$$
$$\Rightarrow x,y,z \text{ form sides of a triangle XYZ with semiperimeter, circumradius and inradius}$$
$$=s,R,r \text{ (say); then : } abc=r^2s, \sum_{cyc}a=s, abc=r^2s, \sum_{cyc}ab=4Rr+r^2,$$
$$\sum_{cyc}a^2=s^2-8Rr-2r^2, \sum_{cyc}a^3=s^3-12Rrs, \sum_{cyc}a^2b^2=r^2((4R+r)^2-2s^2),$$
$$\sum_{cyc}a^3b^3=(4Rr+r^2)^3-12Rr^3s^2 \text{ and then : } \sum_{cyc}\frac{a(b+c)}{a^2+bc}-1=$$
$$\frac{\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4-abc\sum_{cyc}a^3-\sum_{cyc}a^3b^3+3abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)-2a^2b^2c^2}{\sum_{cyc}a^3b^3+abc\sum_{cyc}a^3+2a^2b^2c^2}$$
$$=\frac{\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)-14a^2b^2c^2-abc\sum_{cyc}a^3-\sum_{cyc}a^3b^3+3abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)}{\sum_{cyc}a^3b^3+abc\sum_{cyc}a^3+2a^2b^2c^2}$$
$$\overset{?}{\geq}\frac{16abc}{(a+b)(b+c)(c+a)}$$
$$\Leftrightarrow \frac{(s^2-8Rr-2r^2)(r^2)((4R+r)^2-2s^2)-14r^4s^2-r^2s(s^3-12Rrs)-}{(4Rr+r^2)^3+12Rr^3s^2+3r^2s^2(4Rr+r^2)}\overset{?}{\geq}\frac{16r^2s}{4Rrs}$$
$$\Leftrightarrow -(3R+4r)s^4+(16R^3+60R^2r+90Rr^2-8r^3)s^2-$$
$$r(192R^4+400R^3r+228R^2r^2+51Rr^3+4r^4)\overset{?}{\geq}0 \quad (1)$$

$$\text{Now, Rouche} \Rightarrow s^2-(m-n)\geq 0 \text{ and } s^2-(m+n)\leq 0, \text{where } m=$$
$$2R^2+10Rr-r^2 \ \&\ n=2(R-2r)\sqrt{R^2-2Rr}\ \therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$$
$$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\leq 0$$
$$\Rightarrow P=-(3R+4r)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\geq 0$$
$$\therefore \text{in order to prove } (1), \text{it suffices to prove : LHS of } (1)\overset{?}{\geq} P$$
$$\Leftrightarrow 4Rr^2s^2(R-2r)^2\overset{?}{\geq}0\rightarrow \text{true}\Rightarrow (1) \text{ is true}$$
$$\therefore \sum_{cyc}\frac{a(b+c)}{a^2+bc}\geq 1+\frac{16abc}{(a+b)(b+c)(c+a)}\ \forall\ a,b,c>0,$$
$$'' = '' \text{ iff } a=b=c \text{ (QED)}$$

2202. If $a,b>0; a+b=2$ then:

$$5(a^2+b^2)+\frac{4(ab+a+b)}{a^2+b^2+1}\geq 14$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

Denote: $S=a+b; P=ab\Rightarrow S=2$

$$2=S=a+b\overset{AM-GM}{\geq}2\sqrt{ab}=2\sqrt{P}\Rightarrow\sqrt{P}\leq 1$$

$$\Rightarrow P\leq 1\Rightarrow P-1\leq 0 \quad (1)$$

$$P\leq 1\Rightarrow 10P\leq 10\Rightarrow 10P-19\leq 10-19=-9<0$$

$$\Rightarrow 10P-19<0 \quad (2)$$

By (1); (2) $\Rightarrow (P-1)(10P-19)\geq 0$ (3)

$$5(a^2+b^2)+\frac{4(ab+a+b)}{a^2+b^2+1}\geq 14\Rightarrow 5(S^2-2P)+\frac{4(P+S)}{S^2-2P+1}-14\geq 0$$

$$\Rightarrow 5(4-2P)+\frac{4(P+2)}{5-2P}-14\geq 0$$

$$20-10P+\frac{4P+8}{5-2P}-14\geq 0$$

$$6-10P\geq\frac{4P+8}{2P-5}\Leftrightarrow 3-5P\geq\frac{2P+4}{2P-5}\Leftrightarrow$$

$$(3-5P)(2P-5)\leq 2P+4\Leftrightarrow 6P-15-10P^2+25P\leq 2P+4$$

$$\Leftrightarrow -10P^2+31P-15-2P-4\leq 0$$

$$-10P^2+29P-19\leq 0$$

$$10P^2-29P+19\geq 0$$

$(P-1)(10P-19)\geq 0$ which is true: (3)

Equality holds for $a=b=1$.

2203. If $a,b,c>0$ and $a+b+c\geq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ then prove that :

$$\sum_{cyc}\sqrt{2a^2+4a+3}-\sum_{cyc}a\geq 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } f(x)=\sqrt{2x^2+4x+3}\ \forall\ x>0\ \text{and then}: f''(x)=\frac{2}{(2x^2+4x+3)^{\frac{3}{2}}}$$

$$>0\therefore f(x)\text{ is convex}\Rightarrow\sum_{cyc}\sqrt{2a^2+4a+3}\overset{\text{Jensen}}{\geq}3.\sqrt{2t^2+4t+3}\left(t=\frac{\sum_{cyc}a}{3}\right)$$

$$\overset{?}{\geq}\sum_{cyc}a+6=3t+6\Leftrightarrow 2t^2+4t+3\overset{?}{\geq}t^2+4t+4\Leftrightarrow t^2\overset{?}{\geq}1\Leftrightarrow\frac{\sum_{cyc}a}{3}\overset{?}{\geq}1$$

$$\Leftrightarrow\sum_{cyc}a\overset{?}{\geq}3\to\text{true}\because\sum_{cyc}a\geq\sum_{cyc}\frac{1}{a}\overset{\text{Bergstrom}}{\geq}\frac{9}{\sum_{cyc}a}\Rightarrow\sum_{cyc}a\geq 3$$

$$\therefore\sum_{cyc}\sqrt{2a^2+4a+3}-\sum_{cyc}a\geq 6\ \forall\ a,b,c>0\ \Big|\ a+b+c\geq\frac{1}{a}+\frac{1}{b}+\frac{1}{c},$$

$$''=''\text{ iff }a=b=c=1\ (\text{QED})$$

2204. If $x,y,z>0, x+y+z=3$ and $\lambda\geq 0$ then :

$$\sum_{cyc}\frac{x}{y(x+\lambda z)}\geq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{x}{y(x+\lambda z)}=\sum_{cyc}\frac{\left(\sqrt{\frac{x}{y}}\right)^2}{x+\lambda z}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}\sqrt{\frac{x}{y}}\right)^2}{\sum_{cyc}x+\lambda\sum_{cyc}x}\overset{\text{AM-GM}}{\geq}$$

$$\frac{\left(3.\sqrt[3]{\sqrt{\frac{x}{y}}.\sqrt{\frac{y}{z}}.\sqrt{\frac{z}{x}}}\right)^2}{(\lambda+1)\left(\sum_{cyc}x\right)}\overset{x+y+z=3}{=}\frac{9}{3(\lambda+1)}\ \text{and so},\sum_{cyc}\frac{x}{y(x+\lambda z)}\geq\frac{3}{\lambda+1}$$

$$\forall\ x,y,z>0\ \Big|\ x+y+z=3\ \text{and}\ \lambda\geq 0,''=''\text{ iff }x=y=z=1\ (\text{QED})$$

2205. If $a,b>0$ then prove that :

$$\frac{2a^3+3b^3}{2a^4+3b^4}+\frac{2b^3+3a^3}{2b^4+3a^4}\leq\frac{4}{a+b}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{2x^3+3y^3}{2x^4+3y^4}=\frac{(2x^3+3y^3)(2x^2+3y^2)}{(2x^4+3y^4)(2x^2+3y^2)}\overset{\text{Reverse CBS}}{\leq}\frac{(2x^3+3y^3)(2x^2+3y^2)}{(2x^3+3y^3)^2}$$

$$=\frac{(2x^2+3y^2)(2x+3y)}{(2x^3+3y^3)(2x+3y)}\overset{\text{Reverse CBS}}{\leq}\frac{(2x^2+3y^2)(2x+3y)}{(2x^2+3y^2)^2}$$

$$\therefore \frac{2x^3+3y^3}{2x^4+3y^4}\leq\frac{2x+3y}{2x^2+3y^2}\rightarrow (1)$$

Substituting $x\equiv a$ and $y\equiv b$ in (1), we get : $\frac{2a^3+3b^3}{2a^4+3b^4}\leq\frac{2a+3b}{2a^2+3b^2}\rightarrow (2)$

Substituting $x\equiv b$ and $y\equiv a$ in (1), we get : $\frac{2b^3+3a^3}{2b^4+3a^4}\leq\frac{2b+3a}{2b^2+3a^2}\rightarrow (3)$

$\therefore$ (2) and (3) $\Rightarrow$ it suffices to prove : $\frac{2a+3b}{2a^2+3b^2}+\frac{2b+3a}{2b^2+3a^2}\overset{?}{\leq}\frac{4}{a+b}$

$$\Leftrightarrow 4(2a^2+3b^2)(2b^2+3a^2)\overset{?}{\geq}(a+b)\left(\begin{matrix}(2a+3b)(2b^2+3a^2)+\\(2b+3a)(2a^2+3b^2)\end{matrix}\right)$$

$$\Leftrightarrow 12t^4-25t^3+26t^2-25t+12\overset{?}{\geq}0\ \left(t=\frac{a}{b}\right)$$

$$\Leftrightarrow (t-1)^2((t-1)^2+11t^2+t+11)\overset{?}{\geq}0\rightarrow \text{true}\ \because t>0$$

$$\therefore \frac{2a^3+3b^3}{2a^4+3b^4}+\frac{2b^3+3a^3}{2b^4+3a^4}\leq\frac{4}{a+b}\ \forall\ a,b>0,''='' \text{ iff } a=b\ (\text{QED})$$

2206. *Let a,b,c be positive real numbers such that $a+b+c=3M$.*

Prove that the following inequality holds:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\sqrt{\frac{a+1}{2(b+c+2)}}+\sqrt{\frac{b+1}{2(c+a+2)}}+\sqrt{\frac{c+1}{2(a+b+2)}}\geq 3$$

Proposed by Tran Nam Hai-Vietnam

Solution by Jenish Rijal-Bardiya-Nepal

Here, $$\sqrt{\frac{a+1}{2(b+c+2)}}=\frac{a+1}{\sqrt{2(a+1)(b+c+2)}}\overset{AM-GM}{\geq}\frac{2(a+1)}{2a+b+c+4}$$

Let $S:=a+b+c=3M$. Thus it suffices to prove that:

$$\sum_{cyc}\left(\frac{a}{S-a}+\frac{2a+2}{S+a+4}\right)\geq 3\Leftrightarrow\sum_{cyc}\left(\frac{a}{S-a}+\frac{2a+2}{S+a+4}-1\right)\geq 0$$

$$\textbf{Notice that: } \frac{a}{S-a}+\frac{2a+2}{S+a+4}-1=\frac{(S+2)(2a-b-c)}{(S-a)(S+a+4)}$$

$$\sum_{cyc}\left(\frac{a}{S-a}+\frac{2a+2}{S+a+4}-1\right)=\sum_{cyc}\frac{(S+2)(2a-b-c)}{(S-a)(S+a+4)}=$$

$$=\sum_{cyc}(a-b)(S+2)\left(\frac{1}{(S-a)(S+a+4)}-\frac{1}{(S-b)(S+b+4)}\right)$$

$$=\sum_{cyc}(a-b)(S+2)\left[\frac{(S-b)(S+b+4)-(S-a)(S+a+4)}{(S-a)(S-b)(S+a+4)(S+b+4)}\right]$$

$$=\sum_{cyc}(a-b)(S+2)\left[\frac{(a-b)(a+b+4)}{(S-a)(S-b)(S+a+4)(S+b+4)}\right]=$$

$$=\sum_{cyc}\frac{(a-b)^2(S+2)(a+b+4)}{(S-a)(S-b)(S+a+4)(S+b+4)}\geq 0$$

$$\textit{Equality holds if and only if } a=b=c=M.$$

2207. *Let a,b,c be non − negative real numbers such that $a+b+c=3$.*
Find the smallest value of K so that: $K(1+abc)\geq a^2b+b^2c+c^2a$.

Proposed by Nguyen Van Hoa-Vietnam

Solution by Jenish Rijal-Bardiya-Nepal

$$\textit{Take } a=2,b=1 \textit{ and } c=0. \textit{So our inequality becomes:}$$
$$K(1+2\cdot 1\cdot 0)\geq 2^2\cdot 1+1^2\cdot 0+0^2\cdot 2 \implies K\geq 4$$
Now, let's prove that the smallest value of K is 4.
$$\textit{So we have to prove that: } 4(1+abc)\geq a^2b+b^2c+c^2a$$
Since the inequality is invariant under cyclic permutation, we may assume W.L.O.G. that b is the median(mid − value) of a, b, and c (i.e., b is between a and c).
$$\textit{We have: } a^2b+b^2c+c^2a+abc=b(a+c)^2-c(a-b)(b-c)$$
$$\textit{Notice that: } (a-b)(b-c)\geq 0 \implies c(a-b)(b-c)\geq 0$$
$$\therefore\ a^2b+b^2c+c^2a+abc=b(a+c)^2-c(a-b)(b-c)\leq b(a+c)^2$$
$$b(a+c)^2=4\cdot b\left(\frac{a+c}{2}\right)\left(\frac{a+c}{2}\right)\overset{AM-GM}{\leq} 4\left(\frac{b+\frac{a+c}{2}+\frac{a+c}{2}}{3}\right)^3=$$

$$= 4\left(\frac{a+b+c}{3}\right)^3 = 4\left(\frac{3}{3}\right)^3 = 4$$

$$\therefore a^2b + b^2c + c^2a + abc \leq b(a+c)^2 \leq 4$$

$$\overset{0\leq 5abc}{\Leftrightarrow} a^2b + b^2c + c^2a \leq 4 - abc \overset{\sum}{\leq} 4 - abc + 5abc = 4(1+abc)$$

Hence, the smallest value of K is 4.

2208. If $a, b, c, d, e > 0$ then:

$$\frac{a+b}{c+d+e} + \frac{b+c}{d+e+a} + \frac{d+c}{b+e+a} + \frac{d+e}{a+b+c} + \frac{e+a}{d+b+c} \geq \frac{10}{3}$$

Proposed by Dorin Mărghidanu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{a+b}{c+d+e} + \frac{b+c}{d+e+a} + \frac{d+c}{b+e+a} + \frac{d+e}{a+b+c} + \frac{e+a}{d+b+c} =$$

$$= \left(1 + \frac{a+b}{c+d+e}\right) + \left(1 + \frac{b+c}{d+e+a}\right) + \left(1 + \frac{d+c}{b+e+a}\right) +$$

$$+ \left(1 + \frac{d+e}{a+b+c}\right) + \left(1 + \frac{e+a}{d+b+c}\right) - 5 =$$

$$= (a+b+c+d+e)\left(\frac{1}{c+d+e} + \frac{1}{d+e+a} + \frac{1}{b+e+a} + \frac{1}{a+b+c} + \frac{1}{d+b+c}\right)$$

$$\overset{Bergstrom}{-} 5 \overset{\sum}{\geq}$$

$$\geq (a+b+c+d+e).\frac{(1+1+1+1+1)^2}{3(a+b+c+d+e)} - 5 = \frac{25}{3} - 5 = \frac{10}{3}$$

$$Equality\ holds\ for: a = b = c = d = e.$$

2209. *If $x_1, x_2, x_3 \ldots x_n > 0$ and $x_{n+1} = x_1$, $x_1 + x_2 + x_3 + \cdots + x_n = 1$ then:*

$$\sum_{i=1}^{n} \frac{x_i^3}{3x_i + 2x_{i+1}} \geq \frac{1}{5n}$$

Proposed by Gheorghe Crăciun-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{i=1}^{n} \frac{x_i^3}{3x_i + 2x_{i+1}} \overset{HOLDER}{\geq} \frac{(\sum x_i)^3}{n^{3-2}(\sum_{i=1}^{n} 3x_i + \sum_{i=1}^{n} 2x_{i+1})} = \frac{1}{n(3+2)} = \frac{1}{5n}$$

$$Equality\ holds\ for: x_1 = x_2 = x_3 = \cdots = x_n$$

2210. *If* $x_1, x_2, x_3 \ldots x_n > 0,\ n \geq 3$ *then* :

$$\sum_{cyc} \frac{x_1 + x_2}{x_3 + x_4 + \cdots + x_n} \geq \frac{2n}{n-2}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$S = x_1 + x_2 + x_3 \ldots + x_n$$

$$\sum_{cyc}\left(\frac{x_1 + x_2}{x_3 + x_4 + \cdots + x_n} + 1\right) - n = \sum_{cyc} \frac{S}{S - (x_1 + x_2)} - n =$$

$$S.\left(\sum_{cyc} \frac{1^2}{S - (x_1 + x_2)}\right) - n \overset{Bergstrom}{\geq} S.\frac{(1 + 1 + 1 \ldots + 1)^2}{\sum_{cyc} S - \sum_{cyc}(x_1 + x_2)} - n =$$

$$= S.\frac{n^2}{nS - 2S} - n = \frac{n^2}{n-2} - n = \frac{2n}{n-2}$$

$$Equality\ holds\ for: \quad x_1 = x_2 = x_3 \ldots = x_n$$

2211. Let a, b > 0 and n ∈ ℕ such that $a^{3n} + b^{3n} = 2$. Prove that:

$$(a^{2n} + b^{2n} + 1)(a^{6n} + b^{6n} + 1) \geq \frac{27}{a^n + b^n + 1}$$

Proposed by Marin Chirciu-Romania

Solution by Adgozel Adgozalov-Azerbaijan

$$(a^{2n} + b^{2n} + 1)(a^{6n} + b^{6n} + 1) \geq_{(CS)} (a^{4n} + b^{4n} + 1)^2 \geq_{(CS)} [(a^{3n} + b^{3n} + 1)^2/(a^{2n} + b^{2n} + 1)]^2$$

$$= 81/(a^{2n} + b^{2n} + 1)^2 \geq_{(AM\text{-}GM)} 81/9 = 9 \geq 27/(a^n + b^n + 1)$$

Note: By AM-GM inequality:

$$2a^{3n} + 1 \geq 3a^{2n},\quad 2b^{3n} + 1 \geq 3b^{2n} \Rightarrow 2(a^{3n} + b^{3n}) + 2 \geq 3(a^{2n} + b^{2n}) \Rightarrow a^{2n} + b^{2n} + 1 \leq 3$$

$$a^{3n} + 2 \geq 3a^n,\quad b^{3n} + 2 \geq 3b^n \Rightarrow a^{3n} + b^{3n} + 4 \geq 3(a^n + b^n) \Rightarrow a^n + b^n + 1 \leq 3$$

$$\Rightarrow 81/(a^{2n} + b^{2n} + 1)^2 \geq 81/3^2 = 9 \quad \text{and} \quad 27/(a^n + b^n + 1) \leq 27/3 = 9$$

2212. *If* $a,b,c,x,y,z>0,\ \sum a=\sum x$ *and* $\lambda\geq 0$ *then*:

$$\sum_{cyc}\frac{a^2}{\sqrt{\lambda x^2+xy}}\geq\frac{1}{\sqrt{\lambda+1}}\sum x$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{a^2}{\sqrt{\lambda x^2+xy}}\overset{Bergstrom}{\geq}\frac{(\sum a)^2}{\sum\sqrt{\lambda x^2+xy}}\geq\frac{(\sum x)^2}{\sqrt{3\sum(\lambda x^2+xy)}}\geq\frac{(\sum x)^2}{\sqrt{3(\sum\lambda x^2+\sum xy)}}=$$

$$=\frac{(\sum x)^2}{\sqrt{3(\lambda\sum xy+\sum xy)}}=\frac{1}{\sqrt{\lambda+1}}\cdot\frac{(\sum x)^2}{\sqrt{3\sum xy}}=\frac{1}{\sqrt{\lambda+1}}\sum x$$

$$Equality\ holds\ for:\ \ a=b=c=x=y=z.$$

2213. If $a,b,c>0, a+b+c=1, \lambda-n=1, \lambda\geq 3$, then :

$$\sum_{cyc}a^2\leq\lambda.\sqrt{\frac{\sum_{cyc}a^2}{3}}-n.\sqrt{3abc}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\lambda.\sqrt{\frac{\sum_{cyc}a^2}{3}}-n.\sqrt{3abc}\geq\frac{\lambda}{3}.\sum_{cyc}a-(\lambda-1).\sqrt{3abc\left(\sum_{cyc}a\right)}$$

$$\left(\because\sum_{cyc}a=1\ and\ \because n=\lambda-1\right)\geq\frac{\lambda}{3}.\sum_{cyc}a-(\lambda-1).\sqrt{\left(\sum_{cyc}ab\right)^2}$$

$$(\because\lambda-1\geq 2>0)=\frac{\lambda}{3}.\left(\sum_{cyc}a\right)^2-(\lambda-1)\left(\sum_{cyc}ab\right)\left(\because\sum_{cyc}a=1\right)$$

$$=\frac{\lambda}{3}.\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)+\sum_{cyc}ab\geq\frac{3}{3}.\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)+\sum_{cyc}ab$$

$$\left(\because\ \lambda\geq 3\ and\ \because\sum_{cyc}a^2-\sum_{cyc}ab\geq 0\right)=\sum_{cyc}a^2\ \ and\ so,$$

$$\sum_{cyc} a^2 \leq \lambda.\sqrt{\frac{\sum_{cyc} a^2}{3}} - n.\sqrt{3abc}\ \forall a,b,c>0 \mid a+b+c=1, \lambda - n = 1, \lambda \geq 3,$$

$$'' = '' \text{ iff } a=b=c=\frac{1}{3} \text{ (QED)}$$

2214. *If $x,y>0$ and $x^2+y^2=2$ then:*

$$\frac{1}{\sqrt{2x^2+5x+2}}+\frac{1}{\sqrt{2y^2+5y+2}} \geq \frac{2}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$2=x^2+y^2=\frac{x^2}{1}+\frac{y^2}{1} \overset{Radon}{\geq} \frac{(x+y)^2}{2}\ or, 4\geq (x+y)^2\ or, x+y\leq 2\ (1)$$

$$\frac{1}{\sqrt{2x^2+5x+2}}+\frac{1}{\sqrt{2y^2+5y+2}}=\frac{1}{\sqrt{(x+2)(2x+1)}}+\frac{1}{\sqrt{(y+2)(2y+1)}} \overset{AM-GM}{\geq}$$

$$\geq \frac{2}{(x+2)+(2x+1)}+\frac{2}{(y+2)+(2y+1)}=\frac{2}{3}\left(\frac{1}{x+1}+\frac{1}{y+1}\right)\geq$$

$$\overset{Bergstrom}{\geq} \frac{2}{3}\cdot\frac{(1+1)^2}{x+y+2} \overset{(1)}{\geq} \frac{2}{3}\cdot\frac{4}{2+2}=\frac{2}{3}$$

Equality holds for x=y=1

2215. If $x,y,z>0$ and $x+y+z+2=xyz$ then prove that :

$$\frac{1}{\sqrt{x^3+1}}+\frac{1}{\sqrt{y^3+1}}+\frac{1}{\sqrt{z^3+1}}\geq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x=A+1, y=B+1, z=C+1$ and then : $A+B+C+5=$
$(A+1)(B+1)(C+1) \Rightarrow ABC+AB+BC+CA=4$

$$\Rightarrow \sum_{cBc}\big((2+B)(2+C)\big)-(2+A)(2+B)(2+C)=4-(AB+BC+CA+ABC)$$

$$=0 \therefore \sum_{cBc}\frac{1}{2+A}=1 \rightarrow (m)\ (\because 2+A=x+1>0 \text{ and analogs}) \text{ and setting :}$$

$$\frac{1}{2+A}=\frac{1}{2}-\alpha\Rightarrow A=\frac{2\alpha}{\frac{1}{2}-\alpha}\to(1);\ \text{Similarly, setting}:\frac{1}{2+B}=\frac{1}{2}-\beta\ \text{and}$$

$$\frac{1}{2+C}=\frac{1}{2}-\gamma\Rightarrow 1\overset{\text{via (m)}}{=}\frac{1}{2}-\alpha+\frac{1}{2}-\beta+\frac{1}{2}-\gamma\Rightarrow\alpha+\beta+\gamma=\frac{1}{2}\to(i)$$

$$\beta+\gamma=\frac{1}{2}-\alpha=\frac{1}{2+A}>0\ ;\ \text{similarly},\gamma+\alpha>0\ \&\ \alpha+\beta>0\ \therefore(1),(i)\Rightarrow$$

$$A=\frac{2\alpha}{\beta+\gamma}\ \text{and analogously},B=\frac{2\beta}{\gamma+\alpha}\ \text{and}\ C=\frac{2\gamma}{\alpha+\beta}\ \text{and via such}$$

$$\text{substitutions},x=A+1=\frac{2\alpha}{\beta+\gamma}+1=\frac{(\gamma+\alpha)+(\alpha+\beta)}{\beta+\gamma}=\frac{b+c}{a}$$

$$(a=\beta+\gamma>0;b=\gamma+\alpha>0;c=\alpha+\beta>0);\ \text{similarly},y=\frac{c+a}{b}\ \text{and}\ z=\frac{a+b}{c}$$

$$\text{Assigning } b+c=X,c+a=Y,a+b=Z\Rightarrow X+Y-Z=2c>0,Y+Z-X=2a>0\ \&\ Z+X-Y=2b>0\Rightarrow X+Y>Z,Y+Z>X,Z+X>Y\Rightarrow X,Y,Z\ \text{form sides}$$

of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say);

$$\text{then}:\sum_{cyc}a=s,abc=r^2s,\sum_{cyc}ab=4Rr+r^2,\sum_{cyc}a^2=s^2-8Rr-2r^2,$$

$$\sum_{cyc}a^3=s^3-12Rrs,\sum_{cyc}a^2b^2=r^2((4R+r)^2-2s^2)$$

$$\therefore\sum_{cyc}\frac{1}{\sqrt{x^3+1}}=\sum_{cyc}\left(\frac{1}{\sqrt{\frac{b+c}{a}+1}}\cdot\frac{1}{\sqrt{\left(\frac{b+c}{a}\right)^2-\frac{b+c}{a}+1}}\right)$$

$$=\frac{1}{\sqrt{\sum_{cyc}a}}\cdot\sum_{cyc}\frac{a.\sqrt{a}.a^2.\sqrt{a}}{a^2.\sqrt{a}.\sqrt{(b+c)^2-a(b+c)+a^2}}$$

$$\overset{\text{Bergstrom}}{\geq}\frac{1}{\sqrt{\sum_{cyc}a}}\cdot\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}\left(a^2.\sqrt{a}.\sqrt{(b+c)^2-a(b+c)+a^2}\right)}$$

$$\overset{\text{CBS}}{\geq}\frac{1}{\sqrt{\sum_{cyc}a}}\cdot\frac{\left(\sum_{cyc}a^2\right)^2}{\sqrt{\sum_{cyc}a^3}.\sqrt{2\sum_{cyc}a^2b^2+2abc\sum_{cyc}a-\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)+abc\sum_{cyc}a+\left(\sum_{cyc}a^2\right)^2-2\sum_{cyc}a^2b^2}}\overset{?}{\geq}1$$

$$\Leftrightarrow(s^2-8Rr-2r^2)^4\overset{?}{\geq}s(s^3-12Rrs)\left(\begin{matrix}3r^2s^2+\\(s^2-8Rr-2r^2)(s^2-12Rr-3r^2)\end{matrix}\right)$$

$$\Leftrightarrow-3s^6+(24R^2+60Rr+9r^2)s^4-r(448R^3+480R^2r+156Rr^2+16r^3)s^2+$$

$$8r^2(4R+r)^4\overset{?}{\geq}0\ \text{and}\ \because P=-3s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$

$$\overset{\text{Double-Rouche}}{\geq}0\ \therefore\ \text{to prove}\ (1),\ \text{suffices to prove}:LHS_{(1)}\overset{?}{\geq}P\Leftrightarrow(12R^2+15r^2)s^4-$$

$$r(256R^3+336R^2r+120Rr^2+13r^3)s^2+8r^2(4R+r)^4 \overset{?}{\geq} \textcircled{2}\ 0 \text{ and } \because$$

$$(12R^2+15r^2)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{to prove } \textcircled{1}, \text{it suffices to prove :}$$

$$\text{LHS of } \textcircled{2} \overset{?}{\geq} (12R^2+15r^2)(s^2-16Rr+5r^2)^2 \Leftrightarrow$$

$$(X=128R^3-456R^2r+360Rr^2-163r^3)s^2 \overset{?}{\geq} \textcircled{3}$$

$$r(1024R^4-3968R^3r+3372R^2r^2-2528Rr^3+367r^4)$$

Case 1 $X \geq 0$ and then : LHS of ③ $\overset{\text{Gerretsen}}{\geq} X(16Rr-5r^2) \overset{?}{\geq}$ RHS of ③

$$\Leftrightarrow 256t^4-992t^3+1167t^2-470t+112 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(256t^3-480t^2+207t-56) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow \textcircled{3} \text{ is true}$$

Case 2 $X < 0$ and then : LHS of ③ $\overset{\text{Gerretsen}}{\geq} X(4R^2+4Rr+3r^2) \overset{?}{\geq}$ RHS of ③

$$\Leftrightarrow 128t^5-584t^4+992t^3-988t^2+739t-214 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\left((t-2)(128t^3-72t^2+192t+68)+243\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$\Rightarrow$ ③ is true $\therefore$ combining both cases, ③ $\Rightarrow$ ② $\Rightarrow$ ① is true $\forall \Delta XYZ_{s,R,r}$

$$\therefore \frac{1}{\sqrt{x^3+1}}+\frac{1}{\sqrt{y^3+1}}+\frac{1}{\sqrt{z^3+1}} \geq 1 \ \forall x,y,z>0 \,|\, x+y+z+2=xyz,$$

$$'' = '' \text{ iff } x=y=z=2 \text{ (QED)}$$

2216. *If $x,y>0$ and $xy=4$ then prove that:*

$$\frac{1}{\sqrt{x^3+1}}+\frac{1}{\sqrt{y^3+1}} \geq \frac{2}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$x^{\frac{3}{2}}+y^{\frac{3}{2}}=\frac{x^{\frac{3}{2}}}{(1)^{\frac{1}{2}}}+\frac{y^{\frac{3}{2}}}{(1)^{\frac{1}{2}}} \overset{Radon}{\geq} \frac{(x+y)^{\frac{3}{2}}}{(1+1)^{\frac{1}{2}}} \overset{x+y=t}{=} \frac{t^{\frac{3}{2}}}{\sqrt{2}} \ (1) \ \& \ t=x+y \geq 2\sqrt{xy} \overset{xy=4}{=} 4 \ (2)$$

$$\sqrt{x^3+1}+\sqrt{y^3+1}=\sqrt{\left(x^{\frac{3}{2}}\right)^2+(1)^2}+\sqrt{\left(y^{\frac{3}{2}}\right)^2+(1)^2} \overset{Minkowski}{\geq}$$

$$\geq \sqrt{\left(x^{\frac{3}{2}}+y^{\frac{3}{2}}\right)^2+(1+1)^2} \overset{(1)}{\geq} \sqrt{\frac{t^3}{2}+4}$$

$$\sqrt{x^3+1}.\sqrt{y^3+1}=\sqrt{(x^3+1)(y^3+1)}=\sqrt{1+x^3+y^3+(xy)^3}=$$

$$=\sqrt{1+(x+y)^3-3xy(x+y)+(xy)^3} \overset{\substack{xy=4\\x+y=t}}{=} \sqrt{65-12t+t^3}$$

$$\frac{1}{\sqrt{x^3+1}}+\frac{1}{\sqrt{y^3+1}}=\frac{\sqrt{x^3+1}+\sqrt{y^3+1}}{\sqrt{x^3+1}.\sqrt{y^3+1}}\overset{(1)}{\geq}\frac{\sqrt{\frac{t^3}{2}+4}}{\sqrt{65-12t+t^3}}$$

$$We\ need\ to\ show:\ \frac{\sqrt{\frac{t^3}{2}+4}}{\sqrt{65-12t+t^3}}\geq\frac{2}{3}\ or,3\sqrt{\frac{t^3}{2}+4}\ \geq 2\sqrt{65-12t+t^3}$$

$$or,9\left(\frac{t^3}{2}+4\right)\geq 4(65-12t+t^3)\ or,t^3+96t-448\geq 0$$

$$or,(t-4)(t^2+4t+112)\geq 0\ true\ as\ t\geq 4\ by\ (2)$$

Equality holds for x=y=2

2217. $If\ a,b,c>0,a+b+c=3\ then:$

$$\sum\frac{a}{b}+\sum\frac{1}{\sqrt[3]{a^3+b^3+abc}}\geq 3+\sqrt[3]{9}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\frac{a}{b}=\sum\frac{a^2}{ab}\geq\frac{(a+b+c)^2}{ab+bc+ca}\overset{q=ab+bc+ac}{=}\frac{9}{q}\ (1)$$

$$q=ab+bc+ca\leq\frac{(a+b+c)^2}{3}\overset{a+b+c=3}{=}3\ \&$$

$$q^2=(ab+bc+ca)^2\geq 3abc(a+b+c)=9abc\overset{r=abc}{=}9r\ or,\frac{q^2}{3}\geq r(2)$$

$$Lemma:\forall\ x>0\ ,\frac{1}{\sqrt[3]{x}}\geq\frac{1}{\sqrt[3]{3}}-\frac{x-3}{3^{\frac{7}{3}}}$$

$$Proof:Let\ t=\sqrt[3]{\frac{x}{3}}>0\ then\ x=3t^3$$

$$so,\frac{1}{\sqrt[3]{x}}\geq\frac{1}{\sqrt[3]{3}}-\frac{x-3}{3^{\frac{7}{3}}}or,\frac{1}{\sqrt[3]{3}t}\geq\frac{1}{\sqrt[3]{3}}-\frac{3t^3-3}{3^{\frac{7}{3}}}\ or,\frac{1}{t}\geq 1-\frac{t^3-1}{3}$$

$$or,t^4-4t+3\geq 0\ or,(t-1)^2(t^2+2t+3)\geq 0\ true$$

$$2(a^3+b^3+c^3)+3abc-9$$

$$=2((a+b+c)^3-3(a+b+c)(ab+bc+ca)+3abc))\ +3abc-9$$

$$=9(5-2q+r)\ (putting\ ,q=ab+bc+ca,r=abc,a+b+c=3)$$

$$\sum\frac{1}{\sqrt[3]{a^3+b^3+abc}}\overset{lemma}{\geq}\frac{3}{\sqrt[3]{3}}-\frac{2(a^3+b^3+c^3)+3abc-9}{3^{\frac{7}{3}}}=\frac{3}{\sqrt[3]{3}}-\frac{9(5-2q+r)}{3^{\frac{7}{3}}}$$

$$L.H.S-\left(3+\sqrt[3]{9}\right)=$$

$$=\frac{9}{q}+\left(\frac{3}{\sqrt[3]{3}}-\frac{9(5-2q+r)}{3^{\frac{7}{3}}}\right)-\left(3+\sqrt[3]{9}\right)\overset{\frac{q^2}{3}\geq r}{\geq}\frac{3(3-q)}{q}-\frac{\left(45-18+q^2\right)3^{\frac{2}{3}}}{27}$$

$$=\frac{(3-q)\left(81+(q-15)3^{\frac{2}{3}}\right)}{27q}\geq 0\ \left(as\ q\leq 3\ so\ 3-q>0\ and\ 81-(q-15)3^{\frac{2}{3}}>0\right)$$

$$soL.H.S-\left(3+\sqrt[3]{9}\right)\geq 0\ or,\ \sum\frac{a}{b}+\sum\frac{1}{\sqrt[3]{a^3+b^3+abc}}\geq 3+\sqrt[3]{9}$$

Equality holds for a=b=c=1.

2218. If $a,b,c>0$ then prove that :

$$\sum_{cyc}\frac{(a+b)^3}{c}\geq\sum_{cyc}\frac{(b+c)^3}{c}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{(a+b)^3}{c}\overset{?}{\geq}\sum_{cyc}\frac{(b+c)^3}{c}$$

$$\Leftrightarrow\sum_{cyc}\frac{a^3}{c}+\sum_{cyc}\frac{b^3}{c}+3\sum_{cyc}\frac{a^3b^2+a^2b^3}{abc}\overset{?}{\geq}\sum_{cyc}\frac{b^3}{c}+\sum_{cyc}a^2+3\sum_{cyc}a^2+3\sum_{cyc}ab$$

$$\Leftrightarrow\sum_{cyc}\frac{a^3}{c}+\frac{3}{abc}\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2b^2\right)-abc\left(\sum_{cyc}ab\right)\right)\overset{?}{\geq}4\sum_{cyc}a^2+3\sum_{cyc}ab$$

$$and\ \because\sum_{cyc}\frac{a^3}{c}=\sum_{cyc}\frac{a^4}{ca}\overset{Bergstrom}{\geq}\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}ab}\geq\sum_{cyc}a^2\ \therefore\ it\ suffices\ to\ prove:$$

$$\frac{3}{abc}\cdot\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2b^2\right)-3\sum_{cyc}ab\overset{?}{\geq}3\sum_{cyc}a^2+3\sum_{cyc}ab$$

$$\Leftrightarrow\frac{3}{abc}\cdot\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2b^2\right)\overset{?}{\geq}3\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)=3\left(\sum_{cyc}a\right)^2$$

$$\Leftrightarrow \sum_{cyc} a^2b^2 \overset{?}{\geq} abc\left(\sum_{cyc} a\right) \to \text{true} \because A^2+B^2+C^2 \geq AB+BC+CA \text{ with}$$

$$A \equiv ab, B \equiv bc, C \equiv ca \therefore \sum_{cyc} \frac{(a+b)^3}{c} \geq \sum_{cyc} \frac{(b+c)^3}{c} \quad \forall a,b,c>0,$$

$$'' = '' \text{ iff } a=b=c \text{ (QED)}$$

2219. *Let a,b,c and λ be positive real numbers satisfying $a+b+c=3$. Prove that the following inequality holds:*

$$\frac{a^5}{\lambda a+b+1}+\frac{b^5}{\lambda b+c+1}+\frac{c^5}{\lambda c+a+1} \geq \frac{3}{\lambda+2}$$

Proposed by Tran Nam Hai-Vietnam

Solution by Jenish Rijal-Bardiya-Nepal

Lemma: $\sqrt{a^5}+\sqrt{b^5}+\sqrt{c^5} \geq 3$

Proof: $\frac{\sqrt{a^5}+\sqrt{b^5}+\sqrt{c^5}}{3} \overset{\text{Power Mean}}{\geq} \left(\frac{a+b+c}{3}\right)^{\frac{5}{2}} = \left(\frac{3}{3}\right)^{\frac{5}{2}} = 1 \Leftrightarrow$

$$\sqrt{a^5}+\sqrt{b^5}+\sqrt{c^5} \geq 3$$

$$\frac{a^5}{\lambda a+b+1}+\frac{\lambda a+b+1}{(\lambda+2)^2} \overset{AM-GM}{\geq} 2\sqrt{\frac{a^5}{\lambda a+b+1}\cdot\frac{\lambda a+b+1}{(\lambda+2)^2}} = \frac{2\sqrt{a^5}}{\lambda+2}$$

$$\therefore \sum_{cyc} \frac{a^5}{\lambda a+b+1}+\sum_{cyc} \frac{\lambda a+b+1}{(\lambda+2)^2} \geq \sum_{cyc} \frac{2\sqrt{a^5}}{\lambda+2}$$

$$\Leftrightarrow \sum_{cyc} \frac{a^5}{\lambda a+b+1}+\frac{\lambda(a+b+c)+(a+b+c)+3}{(\lambda+2)^2} \geq \frac{2\left(\sqrt{a^5}+\sqrt{b^5}+\sqrt{c^5}\right)}{\lambda+2}$$

$$\Leftrightarrow \sum_{cyc} \frac{a^5}{\lambda a+b+1}+\frac{\lambda(3)+(3)+3}{(\lambda+2)^2} \geq \frac{2\left(\sqrt{a^5}+\sqrt{b^5}+\sqrt{c^5}\right)}{\lambda+2} \overset{\text{Lemma}}{\geq} \frac{2(3)}{\lambda+2} = \frac{6}{\lambda+2}$$

$$\Leftrightarrow \sum_{cyc} \frac{a^5}{\lambda a+b+1}+\frac{3(\lambda+2)}{(\lambda+2)^2} \geq \frac{6}{\lambda+2} \Leftrightarrow \sum_{cyc} \frac{a^5}{\lambda a+b+1}+\frac{3}{\lambda+2} \geq \frac{6}{\lambda+2}$$

$$\frac{a^5}{\lambda a+b+1}+\frac{b^5}{\lambda b+c+1}+\frac{c^5}{\lambda c+a+1} \geq \frac{3}{\lambda+2}$$

Equality holds iff $a=b=c=1$.

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2220. *Let a,b,c be positive real numbers. Prove that*

$$\frac{(a-b)^2}{ab}+\frac{(b-c)^2}{bc}+\frac{(c-a)^2}{ca}\geq 2\sqrt{2}\cdot\frac{(a-b)^2+(b-c)^2+(c-a)^2}{a^2+b^2+c^2}$$

Proposed by Cheang Chantria-Cambodgia

Solution by Jenish Rijal-Bardiya-Nepal

Here, the given inequality can be written as:

$$\sum_{cyc}(a-b)^2\left[\frac{a^2+b^2+c^2}{ab}-2\sqrt{2}\right]\geq 0$$

Let $S_c:=\frac{a^2+b^2+c^2}{ab}-2\sqrt{2}$, $S_b:=\frac{a^2+b^2+c^2}{ca}-2\sqrt{2}$, *and* $S_a:=\frac{a^2+b^2+c^2}{bc}-2\sqrt{2}$

The inequality is then equivalent to: $(a-b)^2S_c+(c-a)^2S_b+(b-c)^2S_a\geq 0$

W.L.O.G. assume that: $c\geq b\geq a$

$$*\ S_c:=\frac{a^2+b^2+c^2}{ab}-2\sqrt{2}\overset{c\geq b}{\geq}\frac{a^2+b^2+b^2}{ab}-2\sqrt{2}\overset{AM-GM}{\geq}\frac{2\sqrt{a^2\cdot 2b^2}}{ab}-2\sqrt{2}=0.$$

$$\therefore S_c>0$$

$$*\ S_b:=\frac{a^2+b^2+c^2}{ca}-2\sqrt{2}\overset{b\geq a}{\geq}\frac{a^2+a^2+c^2}{ca}-2\sqrt{2}\overset{AM-GM}{\geq}\frac{2\sqrt{2a^2\cdot c^2}}{ca}-2\sqrt{2}=0.$$

$$\therefore S_b\geq 0$$

$$*\ S_b+S_a=\frac{a^2+b^2+c^2}{ca}-2\sqrt{2}+\frac{a^2+b^2+c^2}{bc}-2\sqrt{2}=\frac{(a+b)(a^2+b^2+c^2)}{abc}-4\sqrt{2}\overset{AM-GM}{\geq}$$

$$\frac{(2\sqrt{ab})(2ab+c^2)}{abc}-4\sqrt{2}\overset{AM-GM}{\geq}\frac{(2\sqrt{ab})(2\sqrt{2ab\cdot c^2})}{abc}-4\sqrt{2}=4\sqrt{2}-4\sqrt{2}=0.$$

$$\therefore S_b+S_a\geq 0$$

We also have: $(c-a)^2=[(c-b)+(b-a)]^2\geq(c-b)^2=(b-c)^2$

$$(a-b)^2S_c+(c-a)^2S_b+(b-c)^2S_a\overset{(a-b)^2S_c\geq 0}{\geq}(c-a)^2S_b+(b-c)^2S_a\overset{(c-a)^2\geq(b-c)^2\ \&\ S_b\geq 0}{\geq}$$

$$(b-c)^2S_b+(b-c)^2S_a=(b-c)^2(S_b+S_a)\overset{S_b+S_a\geq 0}{\geq}0$$

$$\text{Equality holds for } a=b=c \text{ or } a=b=\frac{c}{\sqrt{2}} \text{ and permutations.}$$

2221. *If* $a,b,c>0$ *and* $a+b+c=3$ *Prove that* :

$$\sum_{cyc}\frac{a^3}{a+b+1}\geq 1$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{a^3}{a+b+1}\overset{CBS}{\geq}\frac{(a+b+c)^3}{3(a+b+1+b+c+1+c+a+1)}=$$

$$=\frac{3^3}{3(2(a+b+c)+3)}=\frac{27}{3(2\cdot 3+3)}=1$$

$$\text{Equality holds if only if}: \ a=b=c=1$$

2222. *If* $a,b,c>0$ *and* $a+b+c\geq 3$ *then* :

$$\frac{a^8}{a^2+nbc}+\frac{b^8}{b^2+nac}+\frac{c^8}{c^2+nab}+\frac{a^2+b^2+c^2}{n+1}\geq\frac{6}{n+1}$$

Proposed by Konstantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$(LHS)_1=\sum_{cyc}\frac{a^8}{a^2+nbc}=\sum_{cyc}\frac{(a^2)^4}{a^2+nbc}\overset{CBS}{\geq}\frac{(\sum a^2)^4}{3^2.(\sum a^2+n\sum bc)}\overset{\sum bc\leq\sum a^2}{\geq}$$

$$\geq\frac{(\sum a^2)^4}{9(n+1)\sum a^2}=\frac{(\sum a^2)^3}{9(n+1)}$$

$$(LHS)_1\geq\frac{(\sum a^2)^3}{9(n+1)}\ ;\ (LHS)_2\geq\frac{\sum a^2}{(n+1)}$$

$$LHS=(LHS)_1+(LHS)_2\geq\frac{(\sum a^2)^3}{9(n+1)}+\frac{\sum a^2}{(n+1)}\overset{AM-GM}{\geq}$$

$$2\sqrt{\frac{(\sum a^2)^3}{9(n+1)}.\frac{\sum a^2}{(n+1)}}=\frac{2}{3(n+1)}\left(\sum a^2\right)^2\geq\frac{2}{3(n+1)}.9=\frac{6}{n+1}$$

$$\therefore\sum_{cyc}a^2\geq\frac{(\sum_{cyc}a)^2}{3}\geq\frac{9}{3}=3\ \rightarrow\ \sum_{cyc}a^2\geq 3$$

$$\text{Equality holds for}: \ a=b=c=1$$

2223. If $a,b,c>0,\sum_{cyc}a=3$ and $n\in\mathbb{N}$ then :

$$\sum_{cyc}\frac{a^3}{b^nc}+\sum_{cyc}a^n\geq 6$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^3}{b^nc}+\sum_{cyc}a^n=\sum_{cyc}\left(\frac{a^3}{b^nc}+b^n\right)\overset{AM-GM}{\geq}2.\sum_{cyc}\sqrt{\frac{a^3}{b^nc}.b^n}=2.\sum_{cyc}\frac{a.\sqrt{a}.\sqrt{a}}{\sqrt{ca}}$$

$$\overset{Bergstrom}{\geq}2.\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}\sqrt{ab}}\overset{CBS}{\geq}2.\frac{\left(\sum_{cyc}a\right)^2}{\sqrt{3\sum_{cyc}ab}}\geq 2.\frac{\left(\sum_{cyc}a\right)^2}{\sqrt{\left(\sum_{cyc}a\right)^2}}=2\sum_{cyc}a=6 \text{ and so,}$$

$$\sum_{cyc}\frac{a^3}{b^nc}+\sum_{cyc}a^n\geq 6\ \forall\ a,b,c>0,\sum_{cyc}a=3 \text{ and } n\in\mathbb{N},$$

$$''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

2224. If $x,y,z>0,\sum_{cyc}xy=1$ and $\lambda\geq 6$ then :

$$\sum_{cyc}\frac{1}{\lambda x^2+1}\geq\frac{9}{\lambda+3}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{\lambda x^2+1}=\sum_{cyc}\frac{\frac{1}{x^2}}{\lambda+\frac{1}{x^2}}\overset{Bergstrom}{\geq}\frac{\left(\sum_{cyc}\frac{1}{x}\right)^2}{3\lambda+\sum_{cyc}\frac{1}{x^2}}\overset{?}{\geq}\frac{9}{\lambda+3}$$

$$\Leftrightarrow(\lambda+3).\sum_{cyc}\frac{1}{x^2}+2(\lambda+3).\sum_{cyc}\frac{1}{xy}\overset{?}{\geq}27\lambda+9\sum_{cyc}\frac{1}{x^2}$$

$$\Leftrightarrow(\lambda-6).\sum_{cyc}\frac{1}{x^2}+2(\lambda+3).\sum_{cyc}\frac{1}{xy}\underset{(*)}{\overset{?}{\geq}}27\lambda$$

$$\text{Now, LHS of } (*)\overset{Bergstrom}{\geq}(\lambda-6).\sum_{cyc}\frac{1}{x^2}+2(\lambda+3).\frac{9}{\sum_{cyc}xy}$$

$$= (\lambda-6).\sum_{cyc}\frac{1}{x^2}+18(\lambda+3)\left(\because \sum_{cyc} xy=1\right)$$

$$\geq (\lambda-6).\sum_{cyc}\frac{1}{xy}+18(\lambda+3)\ (\because \lambda\geq 6) \overset{\text{Bergstrom}}{\geq} (\lambda-6).\frac{9}{\sum_{cyc} xy}+18(\lambda+3)$$

$$= 9(\lambda-6)+18(\lambda+3)\left(\because \sum_{cyc} xy=1\right) = 27\lambda \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\frac{1}{\lambda x^2+1}\geq \frac{9}{\lambda+3}\ \forall\, x,y,z>0\ \Big|\ \sum_{cyc} xy=1 \text{ and } \lambda\geq 6,$$

$$'' = '' \text{ iff } x=y=z=\frac{1}{\sqrt{3}}\ (\text{QED})$$

2225. *If $a,b,c,d>0$, $abcd=1$ and $n\in N$ then:*

$$\sum_{cyc}\frac{1+a^{2n}}{\sqrt{a}}\geq 8$$

Proposed by Marin Chirciu-Romania

Solution by Shirvan Tahirov-Azerbaijan

$$\therefore \sum_{cyc}\frac{1+a^{2n}}{\sqrt{a}}\geq 8$$

$$S=\sum_{cyc}\left(\frac{1}{\sqrt{a}}+\frac{a^{2n}}{\sqrt{a}}\right)\geq \sum_{cyc}\frac{1}{\sqrt{a}}+\sum_{cyc}\frac{a^{2n}}{\sqrt{a}}=S_1+S_2$$

$$S_1=\sum_{cyc}\frac{1}{\sqrt{a}} \overset{AM-GM}{\geq} 4\sqrt[4]{\frac{1}{\sqrt{abcd}}}=4$$

$$S_2=\sum_{cyc}\frac{a^{2n}}{\sqrt{a}} \overset{AM-GM}{\geq} 4\sqrt[4]{\frac{a^{2n}.b^{2n}.c^{2n}.d^{2n}}{\sqrt{abcd}}}\geq 4\sqrt[4]{(abcd)^{2n-\frac{1}{2}}}=4$$

$$S\geq S_1+S_2\geq 4+4\geq 8$$

$$Equality\ holds\ for:\ \ a=b=c=d=1.$$

2226. *If $a,b,c>0$ and $ab+bc+ac=3$ then prove that:*

$$\frac{(2a+b+c)^3}{b+c}+\frac{(2b+a+c)^3}{a+c}+\frac{(2c+b+a)^3}{b+a}\geq 96$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{(2a+b+c)^3}{b+c}+\frac{(2b+a+c)^3}{a+c}+\frac{(2c+b+a)^3}{b+a} \overset{HOLDER}{\geq}$$
$$=\frac{\big(4(a+b+c)\big)^3}{3.2(a+b+c)}=\frac{32(a+b+c)^2}{3}\geq\frac{32.3(ab+bc+ac)}{3}=96$$

$$Equality\ holds\ if\ a=b=c.$$

2227. If $a,b,c>0$ then prove that :

$$\sum_{cyc}\frac{a}{2a+b+c}\geq\frac{ab+bc+ca}{2(a+b+c)}\cdot\sum_{cyc}\frac{1}{a+b}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Assigning } b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0,$$
$$y+z-x=2a>0 \text{ and } z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$$
$$\Rightarrow x,y,z \text{ form sides of } a \text{ triangle XYZ with semiperimeter, circumradius } a \text{nd inradius}$$
$$=s,R,r\ (say);\ \text{then}: \sum_{cyc}a=s\Rightarrow a=s-x, b=s-y, c=s-z,$$
$$\sum_{cyc}ab=4Rr+r^2, \text{and then}: \sum_{cyc}\frac{a}{2a+b+c}-\frac{ab+bc+ca}{2(a+b+c)}\cdot\sum_{cyc}\frac{1}{a+b}$$
$$=\sum_{cyc}\frac{s-x}{y+z}-\frac{4Rr+r^2}{2s}\cdot\frac{s^2+4Rr+r^2}{4Rrs}$$
$$=\frac{1}{2s(s^2+2Rr+r^2)}\cdot\left(\sum_{cyc}\big(x^2(s-x)\big)+\left(\sum_{cyc}xy\right)\left(\sum_{cyc}(s-x)\right)\right)-$$
$$\frac{4Rr+r^2}{2s}\cdot\frac{s^2+4Rr+r^2}{4Rrs}$$
$$=\frac{1}{2s(s^2+2Rr+r^2)}\cdot\begin{pmatrix}2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)\\+s(s^2+4Rr+r^2)\end{pmatrix}$$
$$-\frac{4Rr+r^2}{2s}\cdot\frac{s^2+4Rr+r^2}{4Rrs}=\frac{s^2+8Rr+5r^2}{2(s^2+2Rr+r^2)}-\frac{(4R+r)(s^2+4Rr+r^2)}{8Rs^2}\overset{?}{\geq}0$$
$$\Leftrightarrow(8R^2+6Rr-2r^2)s^2-r(32R^3+32R^2r+10Rr^2+r^3)\overset{?}{\geq}s^4 \quad (*)$$

$$\text{Now, } s^4 \overset{\text{Gerretsen}}{\leq} (4R^2+4Rr+3r^2)s^2 \overset{?}{\leq} \text{LHS of } (*)$$

$$\Leftrightarrow (4R^2+2Rr-5r^2)s^2 \boxed{\overset{?}{\leq}_{(**)}} r(32R^3+32R^2r+10Rr^2+r^3)$$

$$\text{Again, } (4R^2+2Rr-5r^2)s^2 \overset{\text{Gerretsen}}{\geq} (4R^2+2Rr-5r^2)(16Rr-5r^2) \overset{?}{\geq}$$

$$\text{RHS of } (**) \Leftrightarrow 8R^3-5R^2r-25Rr^2+r^3 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(8R^2+11Rr-3r^2) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (**) \Rightarrow (*) \text{ is true and so,}$$

$$\sum_{cyc} \frac{a}{2a+b+c} \geq \frac{ab+bc+ca}{2(a+b+c)} \cdot \sum_{cyc} \frac{1}{a+b} \quad \forall a,b,c>0, '' = '' \text{ iff } a=b=c \text{ (QED)}$$

2228.

$$\text{Let } a,b,c>0 \text{ such that } abc=1; \text{ then prove that :}$$

$$\frac{a^5}{(b+1)(c^2+c+1)} + \frac{b^5}{(c+1)(a^2+a+1)} + \frac{c^5}{(a+1)(b^2+b+1)} \geq \frac{1}{2}$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{a^5}{(b+1)(c^2+c+1)} = \sum_{cyc} \frac{a^6}{a(b+1)(c^2+c+1)} \overset{\text{Bergstrom}}{\geq}$$

$$\frac{\left(\sum_{cyc} a^3\right)^2}{abc\sum_{cyc} a + 3abc + 2\sum_{cyc} ab + \sum_{cyc} a^2b + \sum_{cyc} a}$$

$$\therefore \text{LHS} \overset{(*)}{\geq} \frac{\left(\sum_{cyc} a^3\right)^2}{abc\sum_{cyc} a + 3abc + 2\sum_{cyc} ab + \sum_{cyc} a^2b + \sum_{cyc} a}$$

$$\text{Now, } \left(\sum_{cyc} a^3\right)^2 \overset{AM-GM}{\geq} 3abc\left(\sum_{cyc} a^3\right) \overset{\text{Holder}}{\geq} \frac{abc}{3}\left(\sum_{cyc} a\right)^3 \overset{AM-GM}{\geq} \frac{abc}{3} \cdot \left(3\sqrt[3]{abc}\right)^2 \cdot \left(\sum_{cyc} a\right)$$

$$\overset{abc=1}{=} 3abc\left(\sum_{cyc} a\right) \Rightarrow abc\sum_{cyc} a \boxed{\overset{①}{\leq}} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2 \text{ and again,}$$

$$\left(\sum_{cyc} a^3\right)^2 \overset{AM-GM}{\geq} 9a^2b^2c^2 \overset{abc=1}{=} 9abc \Rightarrow 3abc \boxed{\overset{②}{\leq}} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2$$

$$\text{Also, } \left(\sum_{cyc} a^3\right)^2 \overset{AM-GM}{\geq} 3abc\sum_{cyc} a^3 \overset{abc=1}{=} 3\sum_{cyc} a^3 \overset{\text{Chebyshev}}{\geq} \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) \geq$$

$$\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) \overset{AM-GM}{\geq} \left(3.\sqrt[3]{abc}\right)\left(\sum_{cyc} ab\right) \overset{abc=1}{=} 3\sum_{cyc} ab$$

$$\Rightarrow \sum_{cyc} ab \overset{③}{\leq} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2$$

$$\text{Moreover, } a^3+a^3+b^3 \overset{AM-GM}{\geq} 3a^2b;\ b^3+b^3+c^3 \overset{AM-GM}{\geq} 3b^2c;$$

$$c^3+c^3+a^3 \overset{AM-GM}{\geq} 3c^2a \text{ and via summation, } 3\sum_{cyc} a^3 \geq 3\sum_{cyc} a^2b \text{ and } \because$$

$$\sum_{cyc} a^3 \overset{AM-GM}{\geq} 3abc = 3 \therefore \left(\sum_{cyc} a^3\right)^2 \geq 3\sum_{cyc} a^2b \Rightarrow \sum_{cyc} a^2b \overset{④}{\leq} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2$$

$$\text{Finally, } \sum_{cyc} a \overset{abc=1}{=} abc\sum_{cyc} a \overset{via\ ①}{\leq} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2 \Rightarrow \sum_{cyc} a \overset{⑤}{\leq} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2$$

$$\therefore ①,②,③,④,⑤,(*) \Rightarrow \sum_{cyc} \frac{a^5}{(b+1)(c^2+c+1)} \geq \frac{\left(\sum_{cyc} a^3\right)^2}{\left(\frac{1}{3}+\frac{1}{3}+\frac{2}{3}+\frac{1}{3}+\frac{1}{3}\right)\left(\sum_{cyc} a^3\right)^2}$$

$$= \frac{1}{2} \text{ and so, } \frac{a^5}{(b+1)(c^2+c+1)} + \frac{b^5}{(c+1)(a^2+a+1)} + \frac{c^5}{(a+1)(b^2+b+1)} \geq \frac{1}{2}$$

$$\forall a,b,c>0 \,|\, abc=1, '' = '' \text{ iff } a=b=c=1 \text{ (QED)}$$

2229.

Let $a,b,c>0$ such that $abc=1$; then prove that :

$$\frac{a^6}{(b+c)(c^2+c+1)} + \frac{b^6}{(c+a)(a^2+a+1)} + \frac{c^6}{(a+b)(b^2+b+1)} \geq \frac{1}{2}$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{a^6}{(b+c)(c^2+c+1)} \overset{Bergstrom}{\geq}$$

$$\frac{\left(\sum_{cyc} a^3\right)^2}{\sum_{cyc} ab^2 + \sum_{cyc} ab + 2\sum_{cyc} a + \sum_{cyc} a^3 + \sum_{cyc} a^2} \rightarrow (*)$$

$$a^3+b^3+b^3 \overset{AM-GM}{\geq} 3ab^2;\ b^3+c^3+c^3 \overset{AM-GM}{\geq} 3bc^2;\ c^3+a^3+a^3 \overset{AM-GM}{\geq} 3ca^2$$

$$\text{and via summation, } 3\sum_{cyc} a^3 \geq 3\sum_{cyc} ab^2 \text{ and } \because \sum_{cyc} a^3 \underbrace{\overset{AM-GM}{\geq}}_{(**)} 3abc = 3$$

$$\therefore \left(\sum_{cyc} a^3\right)^2 \geq 3\sum_{cyc} ab^2 \Rightarrow \sum_{cyc} ab^2 \overset{①}{\leq} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2$$

$$\text{Again, } \left(\sum_{cyc} a^3\right)^2 \overset{\text{via } (**)}{\geq} 3\sum_{cyc} a^3 \overset{\text{Chebyshev}}{\geq} \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) \overset{AM-GM}{\geq} \left(3.\sqrt[3]{abc}\right)\left(\sum_{cyc} a^2\right)$$

$$\overset{abc=1}{=} 3\sum_{cyc} a^2 \Rightarrow \sum_{cyc} a^2 \overset{②}{\leq} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2 \text{ and } \because \sum_{cyc} ab \leq \sum_{cyc} a^2$$

$$\therefore \text{via } ②, \sum_{cyc} ab \overset{③}{\leq} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2$$

$$\text{Moreover, } \left(\sum_{cyc} a^3\right)^2 \overset{\text{via } (**)}{\geq} 3\sum_{cyc} a^3 \overset{\text{Holder}}{\geq} \frac{1}{3}\left(\sum_{cyc} a\right)^3 \overset{AM-GM}{\geq} \frac{1}{3}.\left(3.\sqrt[3]{abc}\right)^2.\left(\sum_{cyc} a\right)$$

$$\overset{abc=1}{=} 3\left(\sum_{cyc} a\right) \Rightarrow \sum_{cyc} a \overset{④}{\leq} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2 \text{ and finally, } \left(\sum_{cyc} a^3\right)^2 \overset{\text{via } (**)}{\geq} 3\sum_{cyc} a^3$$

$$\Rightarrow \sum_{cyc} a^3 \overset{⑤}{\leq} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2 \quad \therefore ①, ②, ③, ④, ⑤, (*) \Rightarrow$$

$$\sum_{cyc} \frac{a^6}{(b+c)(c^2+c+1)} \geq \frac{\left(\sum_{cyc} a^3\right)^2}{\left(\frac{1}{3}+\frac{1}{3}+\frac{2}{3}+\frac{1}{3}+\frac{1}{3}\right)\left(\sum_{cyc} a^3\right)^2} = \frac{1}{2} \text{ and so,}$$

$$\frac{a^6}{(b+c)(c^2+c+1)} + \frac{b^6}{(c+a)(a^2+a+1)} + \frac{c^6}{(a+b)(b^2+b+1)} \geq \frac{1}{2}$$

$$\forall a, b, c > 0 \,|\, abc = 1, '' = '' \text{ iff } a = b = c = 1 \text{ (QED)}$$

2230.

Let $a, b, c > 0$ *such that* $abc = 1$; then prove that :

$$\frac{a^7}{(b^2+c^2)(c^3+c+1)} + \frac{b^7}{(c^2+a^2)(a^3+a+1)} + \frac{c^7}{(a^2+b^2)(b^3+b+1)} \geq \frac{1}{2}$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^7}{(b^2+c^2)(c^3+c+1)}=\sum_{cyc}\frac{a^8}{a(b^2+c^2)(c^3+c+1)}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}a^4\right)^2}{P}$$

$$\to(*);P=\sum_{cyc}\left(a(b^2+c^2)(c^3+c+1)\right)=\sum_{cyc}\left(a\left(\sum_{cyc}a^2-a^2\right)(c^3+c+1)\right)$$

$$=\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^3b+\sum_{cyc}ab+\sum_{cyc}a\right)-\sum_{cyc}a^3b^3-\sum_{cyc}ab^3-\sum_{cyc}a^3$$

$$=\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^3b+\sum_{cyc}ab\right)+\sum_{cyc}a^3+\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-$$

$$\left(3abc+\sum_{cyc}a^3b^3\right)-\sum_{cyc}ab^3-\sum_{cyc}a^3\overset{abc=1}{=}$$

$$\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^3b+\sum_{cyc}ab\right)+\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-$$

$$\left(3a^2b^2c^2+\sum_{cyc}a^3b^3\right)-\sum_{cyc}ab^3\overset{\text{Schur}}{\leq}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^3b+\sum_{cyc}ab\right)+$$

$$\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-\left(abc\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc\right)\right)-\sum_{cyc}ab^3\overset{AM-GM}{\leq}$$

$$\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^3b+\sum_{cyc}ab\right)+\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+3-$$

$$3.\sqrt[3]{a^4b^4c^4}\ (\because abc=1)\overset{Vasc}{\leq}\frac{1}{3}\left(\sum_{cyc}a^2\right)^3+\left(\sum_{cyc}a^2\right)^2+3-3\ (\because abc=1)$$

$$\leq\frac{1}{3}\left(\sum_{cyc}a^2\right).3\sum_{cyc}a^4+3\sum_{cyc}a^4\therefore P\leq\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^4\right)+3\sum_{cyc}a^4\to(**)$$

$$\text{Now,}\left(\sum_{cyc}a^4\right)^2\geq\frac{1}{3}\left(\sum_{cyc}a^2\right)^2\left(\sum_{cyc}a^4\right)\overset{AM-GM}{\geq}\frac{1}{3}.3\sqrt[3]{a^2b^2c^2}.\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^4\right)$$

$$\overset{abc=1}{=}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^4\right)\therefore\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^4\right)\overset{①}{\leq}\left(\sum_{cyc}a^4\right)^2 \text{ and again,}$$

$$\left(\sum_{cyc}a^4\right)^2\geq\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^4\right)\overset{AM-GM}{\geq}3\sqrt[3]{a^2b^2c^2}.\left(\sum_{cyc}a^4\right)\overset{abc=1}{=}3\sum_{cyc}a^4$$

$$\therefore 3\sum_{cyc}a^4\overset{②}{\leq}\left(\sum_{cyc}a^4\right)^2\ \therefore ①,②\text{ and }(**)\Rightarrow P\leq 2\left(\sum_{cyc}a^4\right)^2\rightarrow(***)$$

$$\therefore (*)\text{ and }(***)\Rightarrow\sum_{cyc}\frac{a^7}{(b^2+c^2)(c^3+c+1)}\geq\frac{1}{2}\text{ and so,}$$

$$\frac{a^7}{(b^2+c^2)(c^3+c+1)}+\frac{b^7}{(c^2+a^2)(a^3+a+1)}+\frac{c^7}{(a^2+b^2)(b^3+b+1)}\geq\frac{1}{2}$$

$$\forall a,b,c>0\,\big|\,abc=1,''=''\text{ iff }a=b=c=1\text{ (QED)}$$

2231. If $a,b>0, a+b=2$ and $n\in\mathbb{N}$ then :

$$\sum_{cyc}\left(\frac{a^3}{b^4+1}\right)^n\geq\frac{1}{2^{n-1}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } ab=t \text{ and } \because a+b=2\ \therefore a^2+b^2\overset{(*)}{=}4-2t$$

$$\Rightarrow a^3+b^3=(a+b)(a^2+b^2-ab)=2(4-2t-t)\ \therefore a^3+b^3\overset{(**)}{=}2(4-3t);$$

$$a^4+b^4=(a^2+b^2)^2-2a^2b^2\ \therefore a^4+b^4\overset{(***)}{=}(4-2t)^2-2t^2 \text{ and}$$

$$a^6+b^6=(a^2+b^2)(a^4+b^4-a^2b^2)=(4-2t)((4-2t)^2-2t^2-t^2)$$

$$\therefore a^6+b^6\overset{(****)}{=}(4-2t)((4-2t)^2-3t^2) \text{ and now,}$$

$$\frac{a^3}{b^4+1}+\frac{b^3}{a^4+1}\overset{?}{\geq}1\Leftrightarrow a^7+b^7+a^3+b^3\overset{?}{\geq}a^4+b^4+a^4b^4+1$$

$$\Leftrightarrow (a+b)\left((a^6+b^6)-ab(a^4+b^4)+a^2b^2(a^2+b^2)-a^3b^3\right)+a^3+b^3$$

$$\overset{?}{\geq}a^4+b^4+a^4b^4+1\overset{\text{via}(*),(**),(***)\text{ and }(****)}{=}$$

$$2\left((4-2t)((4-2t)^2-3t^2)-t((4-2t)^2-2t^2)+t^2(4-2t)-t^3\right)+2(4-3t)$$

$$\overset{?}{\geq}(4-2t)^2-2t^2+t^4+1\Leftrightarrow -t^4-14t^3+110t^2-214t+119\overset{?}{\geq}0$$

$$\Leftrightarrow (1-t)(t^3+15t^2+95(1-t)+24)\overset{?}{\geq}0$$

$$\rightarrow \text{true} \because 2 = a + b \overset{AM-GM}{\geq} 2\sqrt{t} \Rightarrow 0 < t \leq 1 \therefore \frac{a^3}{b^4+1} + \frac{b^3}{a^4+1} \overset{①}{\geq} 1$$

$$\text{Now, } \sum_{cyc}\left(\frac{a^3}{b^4+1}\right)^n \overset{Holder}{\geq} \frac{1}{2^{n-1}} \cdot \left(\frac{a^3}{b^4+1} + \frac{b^3}{a^4+1}\right)^n \overset{via\ ①}{\geq} \frac{1}{2^{n-1}}$$

$$(\because n \in \mathbb{N} \Rightarrow n \geq 0) \text{ and so, } \sum_{cyc}\left(\frac{a^3}{b^4+1}\right)^n \geq \frac{1}{2^{n-1}} \forall a, b > 0 \,|\, a + b = 2 \text{ and } n \in \mathbb{N},$$

$$'' = '' \text{ iff } a = b = 1 \text{ (QED)}$$

2232. *If* $a, b, c, x, y, z > 0,\ a + b + c = 3$ *and* $\lambda \geq 0$ *then*:

$$\sum_{cyc} \sqrt{1 + \lambda a^2} \geq 3\sqrt{\lambda + 1}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc} \sqrt{1 + \lambda a^2} = \sqrt{1^2 + (\sqrt{\lambda}a)^2} + \sqrt{1^2 + (\sqrt{\lambda}b)^2} + \sqrt{1^2 + (\sqrt{\lambda}c)^2} \overset{Minkowski}{\geq}$$

$$\geq \sqrt{(1+1+1)^2 + \left(\sqrt{\lambda}a + \sqrt{\lambda}b + \sqrt{\lambda}c\right)^2} = \sqrt{9 + \lambda(a+b+c)^2} = 3\sqrt{\lambda + 1}$$

$$Equality\ holds\ for: \quad a = b = c = 1$$

2233. If $a, b, c \geq 0$ *and* $ab + bc + ca > 0$ then prove that :

$$\sum_{cyc} \frac{1}{a^2 + bc} \geq \frac{3(a+b+c)}{(a+b)(b+c)(c+a)}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Case 1 $Exactly$ one among $a, b, c = 0$ and WLOG we may assume $a = 0$ $(b, c > 0)$ and then : $\sum_{cyc} \frac{1}{a^2 + bc} \overset{?}{\geq} \frac{3(a+b+c)}{(a+b)(b+c)(c+a)}$

$$\Leftrightarrow \frac{1}{bc} + \frac{1}{b^2} + \frac{1}{c^2} \overset{?}{\geq} \frac{3(b+c)}{bc(b+c)} \Leftrightarrow \left(\frac{1}{b} - \frac{1}{c}\right)^2 \overset{?}{\geq} 0 \rightarrow \text{true}$$

$$\therefore \sum_{cyc} \frac{1}{a^2 + bc} \geq \frac{3(a+b+c)}{(a+b)(b+c)(c+a)}$$

Case 2 $a, b, c > 0$ and then, assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z$

$= 2c > 0, y + z - x = 2a > 0 \text{and } z + x - y = 2b > 0 \Rightarrow x + y > z,$
$y + z > x, z + x > y \Rightarrow x, y, z \text{ form sides of } a \text{ triangle XYZ with semiperimeter,}$
$\text{circumradius and inradius} = s, R, r \text{ (say); then : } abc = r^2 s, \sum_{cyc} a = s, abc = r^2 s,$

$$\sum_{cyc} ab = 4Rr + r^2, \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2, \sum_{cyc} a^3 = s^3 - 12Rrs,$$

$$\sum_{cyc} a^2 b^2 = r^2((4R + r)^2 - 2s^2), \sum_{cyc} a^3 b^3 = (4Rr + r^2)^3 - 12Rr^3 s^2 \text{ and then :}$$

$$\sum_{cyc} \frac{1}{a^2 + bc} \overset{?}{\geq} \frac{3(a + b + c)}{(a + b)(b + c)(c + a)}$$

$$\Leftrightarrow \frac{(\sum_{cyc} ab)(\sum_{cyc} a^2) + \sum_{cyc} a^2 b^2}{abc \sum_{cyc} a^3 + \sum_{cyc} a^3 b^3 + 2a^2 b^2 c^2} \overset{?}{\geq} \frac{3(a + b + c)}{(a + b)(b + c)(c + a)}$$

$$\Leftrightarrow \frac{(4Rr + r^2)(s^2 - 8Rr - 2r^2) + r^2((4R + r)^2 - 2s^2)}{r^2 s(s^3 - 12Rrs) + (4Rr + r^2)^3 - 12Rr^3 s^2 + 2r^4 s^2} \overset{?}{\geq} \frac{3s}{4Rrs}$$

$$\Leftrightarrow -3s^4 + (16R^2 + 68Rr - 6r^2)s^2 - r(256R^3 + 176R^2 r + 40Rr^2 + 3r^3) \overset{?}{\underset{①}{\geq}} 0$$

$\text{Now, Rouche} \Rightarrow s^2 - (m - n) \geq 0 \text{ and } s^2 - (m + n) \leq 0, \text{where}$
$m = 2R^2 + 10Rr - r^2 \text{ and } n = 2(R - 2r).\sqrt{R^2 - 2Rr}$
$\therefore \left(s^2 - (m + n)\right)\left(s^2 - (m - n)\right) \leq 0 \Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0$
$\Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3 \leq 0$
$\Rightarrow P = -3(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3) \geq 0$
$\therefore \text{in order to prove } ①, \text{it suffices to prove : LHS of } ① \overset{?}{\geq} P$

$$\Leftrightarrow (R + 2r)s^2 \overset{?}{\underset{②}{\geq}} r(4R + r)^2 \text{ and finally, } (R + 2r)s^2 \overset{\text{Gerretsen}}{\geq} (R + 2r)(16Rr - 5r^2)$$

$$\overset{?}{\geq} r(4R + r)^2 \Leftrightarrow r^2(19R - 11r) \overset{?}{\geq} 0 \rightarrow \text{true (strict inequality)} \because R \overset{\text{Euler}}{\geq} 2r$$

$\Rightarrow ② \Rightarrow ① \text{ is true} \therefore \sum_{cyc} \frac{1}{a^2 + bc} > \frac{3(a + b + c)}{(a + b)(b + c)(c + a)} \quad \forall a, b, c > 0 \text{ and so,}$

$$\sum_{cyc} \frac{1}{a^2 + bc} \geq \frac{3(a + b + c)}{(a + b)(b + c)(c + a)} \quad \forall a, b, c \geq 0,$$

$'' = '' \text{ iff } \begin{pmatrix} a = 0 \\ b = c > 0 \end{pmatrix} \text{ or } \begin{pmatrix} b = 0 \\ c = a > 0 \end{pmatrix} \text{ or } \begin{pmatrix} c = 0 \\ a = b > 0 \end{pmatrix} \text{ (QED)}$

2234. If $a, b > 0$ and $a^2 + b^2 = 2$ then prove that:

$$\frac{a^3}{a + 1} + \frac{b^3}{b + 1} \geq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Adgozel Adgozalov-Azerbaijan

$$\frac{a^3}{a+1}+\frac{b^3}{b+1}=\frac{a^4}{a^2+a}+\frac{b^4}{b^2+b}\geq\frac{(a^2+b^2)^2}{a^2+b^2+a+b}=$$

$$=\frac{2^2}{2+a+b}=\frac{4}{2+a+b}\geq\frac{4}{2+2}=1$$

$Note$:

$$a^2+b^2\geq\frac{(a+b)^2}{2}\Longrightarrow 2\geq\frac{(a+b)^2}{2}\Longrightarrow(a+b)^2\leq 4\ \Longrightarrow a+b\leq 2$$

$''=''$ if $a=b=1$

2235. *If* $a,b,c>0$ *then prove that*:

$$\sum_{cyc}\frac{a^2b^2}{a+b}\leq\sum_{cyc}\frac{b^2c^2}{a+b}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{a^2b^2}{a+b}+\frac{c^2b^2}{c+b}+\frac{a^2c^2}{a+c}\leq\frac{c^2b^2}{a+b}+\frac{a^2b^2}{a+c}+\frac{a^2c^2}{b+c}$$

$$Wlog: a\leq b\leq c\ , a^2b^2\leq a^2c^2\leq c^2b^2$$

$$a,b,c>0,\ \begin{cases}a+b\leq a+c\leq b+c\\ \frac{1}{a+b}\geq\frac{1}{a+c}\geq\frac{1}{b+c}\end{cases}$$

According to the conditions of the Rearrangement inequality:

$$B_{min}=\frac{a^2b^2}{a+b}+\frac{c^2b^2}{c+b}+\frac{a^2c^2}{a+c},\qquad C_{mixed}=\frac{c^2b^2}{a+b}+\frac{a^2b^2}{a+c}+\frac{a^2c^2}{b+c}$$

$$So,\ B_{min}\leq C_{mixed}.\ \ Equality\ holds\ for:\ \ a=b=c$$

2236. If $a,b>0; a+b=2$ then:

$$\frac{1}{a^3}+\frac{1}{b^3}+12ab\geq 14$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{Denote: } \begin{cases} S = a+b \\ P = ab \end{cases} \Rightarrow \begin{cases} S = 2 \\ a^3+b^3 = S^3 - 3SP = 8 - 6P \end{cases}$$

$$\frac{1}{a^3} + \frac{1}{b^3} + 12ab \geq 14 \Leftrightarrow \frac{a^3+b^3}{a^3b^3} + 12ab \geq 14$$

$$\Leftrightarrow \frac{S^3 - 3SP}{P^3} + 12P \geq 14 \Leftrightarrow \frac{8-6P}{P^3} + 12P \geq 14$$

$$8 - 6P + 12P^4 \geq 14P^3, \qquad 12P^4 - 14P^3 - 6P + 8 \geq 0$$

$$6P^4 - 7P^3 - 3P + 4 \geq 0, \qquad 6P^4 - 6P^3 - P^3 + P^2 - P^2 + P - 4P + 4 \geq 0$$

$$6P^3(P-1) - P^2(P-1) - P(P-1) - 4(P-1) \geq 0$$

$$(P-1)(6P^3 - P^2 - P - 4) \geq 0$$

$$(P-1)[(6P^3 - 6P^2) + (5P^2 - 5P) + (4P-4)] \geq 0$$

$$(P-1)[6P^2(P-1) + 5P(P-1) + 4(P-1)] \geq 0, \ (P-1)^2(6P^2 + 5P + 4) \geq 0$$

(true)

$a, b > 0 \Rightarrow ab > 0 \Rightarrow P > 0 \Rightarrow 6P^2 + 5P + 4 > 0$ Equality holds for $a = b = 1$

2237. If $a, b > 0; a + b = 2$ then:

$$\frac{1}{a^2} + \frac{1}{b^2} + 6ab \geq 8$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{Denote: } \begin{cases} S = a+b \\ P = ab \end{cases} \Rightarrow \begin{cases} S = 2 \\ a^2+b^2 = S^2 - 2P = 4 - 2P \end{cases}$$

$$\frac{1}{a^2} + \frac{1}{b^2} + 6ab \geq 8 \Leftrightarrow \frac{a^2+b^2}{a^2b^2} + 6ab \geq 8$$

$$\frac{4-2P}{P^2} + 6P \geq 8 \Leftrightarrow 4 - 2P + 6P^3 - 8P^2 \geq 0$$

$$6P^3 - 6P^2 - 2P^2 + 2P - 4P + 4 \geq 0, \qquad 6P^2(P-1) - 2P(P-1) - 4(P-1) \geq 0$$

$$(P-1)(3P^2 - P - 2) \geq 0, \qquad (P-1)(3P^2 - 3P + 2P - 2) \geq 0$$

$$(P-1)[3P(P-1) + 2(P-1)] \geq 0, \ (P-1)^2(3P+2) \geq 0 \quad \text{(true)}$$

$$a, b > 0 \Rightarrow ab > 0 \Rightarrow P > 0 \Rightarrow 3P + 2 > 0$$

Equality holds for $ab = 1 \Rightarrow a = b = 1$.

2238. *If $a, b, c > 0$ are such that $abc = 1$ then prove that:*

$$\frac{a^{15}}{(b^2+c^2)\sqrt{c^3+c+1}} + \frac{b^{15}}{(c^2+a^2)\sqrt{a^3+a+1}} + \frac{c^{15}}{(a^2+b^2)\sqrt{b^3+b+1}} \geq \frac{\sqrt{3}}{2}$$

Proposed by Hikmat Mammadov – Azerbaijan

Solution by Jenish Rijal – Bardiya – Nepal:

$$\text{Here, } \sqrt{3(c^3+c+1)} \overset{AM-GM}{\leq} \frac{3+c^3+c+1}{2} \quad \therefore \sqrt{c^3+c+1} \leq \frac{c^3+c+4}{2\sqrt{3}}$$

$$\text{So, } \sum_{cyc} \frac{a^{15}}{(b^2+c^2)\sqrt{c^3+c+1}} \geq \sum_{cyc} \frac{a^{15}}{(b^2+c^2)(c^3+c+4)} \cdot 2\sqrt{3}$$

$$\text{Hence, it suffices to prove that: } \sum_{cyc} \frac{a^{15}}{(b^2+c^2)(c^3+c+4)} \geq \frac{1}{4}$$

Lemma: If $a, b, c > 0$ such that $abc = 1$ then $\sum a^p \geq \sum a^q$ if $p \geq q \geq 0$.

Proof: W.L.O.G. assume that

$a \geq b \geq c$. The sequences (a^q, b^q, c^q) and $(a^{p-q}, b^{p-q}, c^{p-q})$ are similarly sorted.

Hence, by Chebyshev's inequality,

$$\sum a^p = \sum a^q \cdot a^{p-q} \geq \frac{1}{3}\left(\sum a^q\right)\left(\sum a^{p-q}\right) \overset{AM-GM}{\geq} \frac{1}{3}\left(\sum a^q\right) \cdot 3 \cdot \left(\sqrt[3]{(abc)^{p-q}}\right) =$$

$$= \left(\sum a^q\right) \cdot \left(\sqrt[3]{(1)^{p-q}}\right) = \sum a^q$$

Therefore, by the lemma, we have:

$$\sum a^5 \geq \sum a^3 \geq \sum a^2 \geq \sum a \overset{AM-GM}{\geq} 3\sqrt[3]{abc} = 3$$

$$\text{Now, } \frac{a^{15}}{(b^2+c^2)(c^3+c+4)} + \frac{b^2+c^2}{24} + \frac{c^3+c+4}{72} \overset{AM-GM}{\geq} 3\sqrt[3]{\frac{a^{15}}{1728}} = 3 \cdot \frac{a^5}{12} = \frac{a^5}{4}$$

$$\therefore \sum_{cyc} \frac{a^{15}}{(b^2+c^2)(c^3+c+4)} + \sum_{cyc} \frac{b^2+c^2}{24} + \sum_{cyc} \frac{c^3+c+4}{72} \geq \sum_{cyc} \frac{a^5}{4}$$

$$\Leftrightarrow \sum_{cyc} \frac{a^{15}}{(b^2+c^2)(c^3+c+4)} + \frac{\sum a^2}{12} + \frac{\sum a^3 + \sum a + 12}{72} \geq \sum_{cyc} \frac{a^5}{4}$$

$$\Leftrightarrow \sum_{cyc} \frac{a^{15}}{(b^2+c^2)(c^3+c+4)} \geq \frac{\sum a^5}{4} - \frac{\sum a^3}{72} - \frac{\sum a^2}{12} - \frac{\sum a}{72} - \frac{12}{72} \overset{Lemma}{\geq}$$

$$\geq \frac{\sum a^5}{4} - \frac{\sum a^5}{72} - \frac{\sum a^5}{12} - \frac{\sum a^5}{72} - \frac{1}{6} \geq \frac{5}{36} \cdot \left(\sum a^5\right) - \frac{1}{6} \geq \frac{5}{36} \cdot (3) - \frac{1}{6} = \frac{1}{4}$$

Equality holds if and only if $a = b = c = 1$.

2239. Let a, b, c be positive real numbers such that $ab + bc + ca = 1$ prove that

$$\sqrt{\frac{1}{a^2} + 1} + \sqrt{\frac{1}{b^2} + 1} + \sqrt{\frac{1}{c^2} + 1} \geq 6$$

Proposed by Jenish Rijal-Bardiya-Nepal

Solution by Sahel Koley-Tarakeswar-India

For any $a, b, c > 0$ with $ab + bc + ca = 1$, we can find a triangle ABC such that

$$a = \tan\frac{A}{2}, \quad b = \tan\frac{B}{2}, \quad c = \tan\frac{C}{2}$$

and also $A + B + C = \pi$. Now, we can write that:

$$\sqrt{\frac{1}{a^2} + 1} = \sqrt{\cot^2\frac{A}{2} + 1} = \sqrt{\csc^2\frac{A}{2}} = \csc\frac{A}{2}$$

as $0 < \frac{A}{2} < \frac{\pi}{2}$, so, we have to prove for $A + B + C = \pi$ that

$$\csc\left(\frac{A}{2}\right) + \csc\left(\frac{B}{2}\right) + \csc\left(\frac{C}{2}\right) \geq 6$$

Let, $f(x) = \csc x, \quad 0 < x < \frac{\pi}{2}, \quad f'(x) = -\csc x \cot x,$ f''(x)= cscx (cot²x + csc²x)>0

so f is strictly convex on $\left(0, \frac{\pi}{2}\right)$. Therefore, by Jensen inequality, we get

$$f\left(\frac{A}{2}\right)+f\left(\frac{B}{2}\right)+f\left(\frac{C}{2}\right)\geq 3f\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)$$

$$\Rightarrow \sum \csc\left(\frac{A}{2}\right)\geq 3\csc\left(\frac{A+B+C}{6}\right)\Rightarrow \sum \csc\left(\frac{A}{2}\right)\geq 3\csc\left(\frac{\pi}{6}\right)=3.2=6$$

$$\sqrt{\frac{1}{a^2}+1}+\sqrt{\frac{1}{b^2}+1}+\sqrt{\frac{1}{c^2}+1}\geq 6$$

Equality occurs for $A=B=C=\frac{\pi}{6}$, i.e. $a=b=c=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$

2240. *If* $a,b,c>0$, *and* $\lambda\geq 0$ *then*:

$$\sum_{cyc}\frac{ab}{a+\lambda b}\geq\frac{1}{\lambda+1}\sum_{cyc}a$$

Proposed by Marin Chirciu-Romania

Solution by Shirvan Tahirov-Azerbaijan

$$\sum_{cyc}\frac{ab}{a+\lambda b}=\sum_{cyc}\left(b-\frac{\lambda b^2}{a+\lambda b}\right)=\sum_{cyc}b-\lambda\sum_{cyc}\frac{b^2}{a+\lambda b}\overset{Bergstrom}{\leq}$$

$$\leq\sum_{cyc}a-\lambda\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}a+\lambda\sum_{cyc}a}=\sum_{cyc}a-\frac{\lambda\sum_{cyc}a}{1+\lambda}=\frac{1}{\lambda+1}\sum_{cyc}a$$

$$Equality\ holds\ for:\ a=b=c.$$

2241. If $a,b>0; ab=1$ then:

$$\frac{a}{b^3+1}+\frac{b}{a^3+1}\geq 1$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

Denote: $\begin{cases}S=a+b>0\\ P=ab>0\end{cases}$

$$S=a+b\overset{AM-GM}{\geq}2\sqrt{ab}=2\cdot\sqrt{1}=2\Rightarrow S\geq 2\Rightarrow S-2\geq 0\quad (1)$$

$$\frac{a}{b^3+1}+\frac{b}{a^3+1}\geq 1 \Leftrightarrow a(a^3+1)+b(b^3+1)\geq (a^3+1)(b^3+1)$$

$$a^4+b^4+a+b\geq (ab)^3+a^3+b^3+1$$

$$(S^2-2P)^2-2P^2+S\geq 1+S^3-3SP+1$$

$$(S^2-2)^2-2+S-1-S^3+3S-1\geq 0$$

$$S^4-4S^2+4-4-S^3+4S\geq 0, \qquad S^4-S^3-4S^2+4S\geq 0$$

$$S^3-S^2-4S+4\geq 0, \qquad S^2(S-1)-4(S-1)\geq 0$$

$$S^2-4\geq 0,\ (S-2)(S+2)\geq 0,\ S-2\geq 0 \quad \text{(True by (1))}$$

Equality holds for $a=b=1$.

2242. *Let a,b,c be positive real numbers. Prove that:*

$$\frac{(a-b)^2}{ab}+\frac{(b-c)^2}{bc}+\frac{(c-a)^2}{ca}\geq 2\sqrt{2}\cdot\frac{(a-b)^2+(b-c)^2+(c-a)^2}{a^2+b^2+c^2}$$

Proposed by Cheang Chantria-Cambodgia

Solution by Jenish Rijal-Bardiya-Nepal

Here, the given inequality can be written as:

$$\sum_{cyc}(a-b)^2\left[\frac{a^2+b^2+c^2}{ab}-2\sqrt{2}\right]\geq 0$$

Let $S_c:=\frac{a^2+b^2+c^2}{ab}-2\sqrt{2}$, $S_b:=\frac{a^2+b^2+c^2}{ca}-2\sqrt{2}$, *and*

$$S_a:=\frac{a^2+b^2+c^2}{bc}-2\sqrt{2}$$

The inequality is then equivalent to: $(a-b)^2S_c+(c-a)^2S_b+(b-c)^2S_a\geq 0$

W.L.O.G. assume that: $c\geq b\geq a$

$$*\ S_c:=\frac{a^2+b^2+c^2}{ab}-2\sqrt{2}\overset{c\geq b}{\geq}\frac{a^2+b^2+b^2}{ab}-2\sqrt{2}\overset{AM-GM}{\geq}\frac{2\sqrt{a^2\cdot 2b^2}}{ab}-2\sqrt{2}=0.$$

$$\therefore S_c>0$$

$$*\ S_b:=\frac{a^2+b^2+c^2}{ca}-2\sqrt{2}\overset{b\geq a}{\geq}\frac{a^2+a^2+c^2}{ca}-2\sqrt{2}\overset{AM-GM}{\geq}\frac{2\sqrt{2a^2\cdot c^2}}{ca}-2\sqrt{2}=0.$$

$$\therefore S_b\geq 0$$

$$*\ S_b + S_a = \frac{a^2+b^2+c^2}{ca} - 2\sqrt{2} + \frac{a^2+b^2+c^2}{bc} - 2\sqrt{2} =$$

$$= \frac{(a+b)(a^2+b^2+c^2)}{abc} - 4\sqrt{2} \overset{AM-GM}{\geq} \frac{\left(2\sqrt{ab}\right)(2ab+c^2)}{abc} - 4\sqrt{2} \overset{AM-GM}{\geq}$$

$$\geq \frac{\left(2\sqrt{ab}\right)\left(2\sqrt{2ab\cdot c^2}\right)}{abc} - 4\sqrt{2} = 4\sqrt{2} - 4\sqrt{2} = 0. \quad \therefore S_b + S_a \geq 0$$

We also have:

$$(c-a)^2 = [(c-b)+(b-a)]^2 \geq (c-b)^2 = (b-c)^2$$

Now:

$$(a-b)^2 S_c + (c-a)^2 S_b + (b-c)^2 S_a \overset{(a-b)^2 S_c \geq 0}{\geq}$$

$$\geq (c-a)^2 S_b + (b-c)^2 S_a \overset{(c-a)^2 \geq (b-c)^2 \ \& \ S_b \geq 0}{\geq}$$

$$\geq (b-c)^2 S_b + (b-c)^2 S_a = (b-c)^2 (S_b + S_a) \overset{S_b + S_a \geq 0}{\geq} 0$$

$$\textit{Equality holds for } \{a=b=c\} \textit{ or } \left\{a=b=\frac{c}{\sqrt{2}} \textit{ and permutations}\right\}.$$

Remark: $S_c > 0$ *because the equality is not possible*

in that case due to our assumption $b \geq a$.

2243. ***Let*** a, b, c ***be positive real numbers*** $\& \ \lambda \geq 1$. ***Prove that***:

$$\frac{a}{\lambda\sqrt{a^2+8bc} - (\lambda-1)a} + \frac{b}{\lambda\sqrt{b^2+8ac} - (\lambda-1)b} + \frac{c}{\lambda\sqrt{c^2+8ab} - (\lambda-1)c} \geq \frac{3}{2\lambda+1}$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$a^2 < a^2 + 8bc \ (a,b,c>0) \ or, a < \sqrt{a^2+8bc} \ or, \frac{a}{\sqrt{a^2+8bc}} < 1$$

$$Let\ x = \frac{a}{\sqrt{a^2+8bc}}, y = \frac{b}{\sqrt{b^2+8ac}}, z = \frac{c}{\sqrt{c^2+8ab}}, x,y,z \in (0,1)$$

A famous inequality from IMO 2001 *(problem* 2*), states that for any positive real number* a, b, c *we get*:

$$x+y+z = \frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ac}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1 (1)$$

$$\frac{a}{\lambda\sqrt{a^2+8bc}-(\lambda-1)a}+\frac{b}{\lambda\sqrt{b^2+8ac}-(\lambda-1)b}+\frac{c}{\lambda\sqrt{c^2+8ab}-(\lambda-1)c}$$

$$=\frac{\frac{a}{\sqrt{a^2+8bc}}}{\lambda-(\lambda-1)\left(\frac{a}{\sqrt{a^2+8bc}}\right)}+\frac{\frac{b}{\sqrt{b^2+8ac}}}{\lambda-(\lambda-1)\left(\frac{b}{\sqrt{b^2+8ac}}\right)}+\frac{\frac{c}{\sqrt{c^2+8ab}}}{\lambda-(\lambda-1)\left(\frac{c}{\sqrt{c^2+8ab}}\right)}$$

$$=\frac{x}{\lambda-(\lambda-1)x}+\frac{y}{\lambda-(\lambda-1)y}+\frac{z}{\lambda-(\lambda-1)z}$$

$$Let\ f(t)=\frac{t}{\lambda-(\lambda-1)t}, t\in(0,1)\ \ f'(t)=\frac{\lambda}{(\lambda-(\lambda-1)t)^2}>0\ \&$$

$$f''(t)=\frac{2\lambda(\lambda-1)}{(\lambda-(\lambda-1)t)^3}>0\ (as\ \lambda-(\lambda-1)t=\lambda(1-t)+t>0\ , t<1\ and\ \lambda>1)$$

$$so\ f\ is\ convex\ and\ increasing\ on(0,1)\ ,$$

$$using\ Jensen\ inequaliy\ \ f(x)+f(y)+f(z)\geq 3f\left(\frac{x+y+z}{3}\right)\overset{(1)}{\geq}3f\left(\frac{1}{3}\right)$$

$$\frac{a}{\lambda\sqrt{a^2+8bc}-(\lambda-1)a}+\frac{b}{\lambda\sqrt{b^2+8ac}-(\lambda-1)b}+\frac{c}{\lambda\sqrt{c^2+8ab}-(\lambda-1)c}\geq$$

$$\geq 3\frac{\frac{1}{3}}{\lambda-(\lambda-1)\frac{1}{3}}=\frac{3}{2\lambda+1}$$

Equality holds for a=b=c.

2244. *Let a,b,c positive real numbers with $abc=1$. Prove that:*

$$\sqrt[3]{(a^2+b+c)(b^2+c+a)(c^2+a+b)}\geq\sqrt{\frac{3}{2}\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$(a+b+c)\overset{AM-GM}{\geq}3\sqrt[3]{abc}\overset{abc=1}{=}3\ or, x\overset{a+b+c=x}{\geq}3$$

$$(a^2+b+c)(1+b+c)\overset{C-S}{\geq}(a+b+c)^2\ or, (a^2+b+c)\geq\frac{(a+b+c)^2}{(1+b+c)}$$

$$Similarly: (b^2+c+a)\geq\frac{(a+b+c)^2}{1+c+a}\ \&\ (c^2+a+b)\geq\frac{(a+b+c)^2}{1+a+b}$$

$$\sqrt[3]{(a^2+b+c)(b^2+c+a)(c^2+a+b)}\geq\sqrt[3]{\frac{(a+b+c)^2(b+c+a)^2(c+a+b)^2}{(1+b+c)(1+c+a)(1+a+b)}}$$

$$=\frac{(a+b+c)^2}{\sqrt[3]{(1+b+c)(1+c+a)(1+a+b)}}\overset{AM-GM}{\geq}\frac{(a+b+c)^2}{\frac{(1+b+c)+(1+c+a)+(1+a+b)}{3}}$$

$$=$$

$$=\frac{3(a+b+c)^2}{3+2(a+b+c)}=\frac{3x^2}{3+2x}\ so, L.H.S\geq\frac{3x^2}{3+2x}$$

$$\sqrt{\frac{3}{2}\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}=\sqrt{\frac{3}{2}\left(a+b+c+\frac{ab+bc+ca}{abc}\right)}\overset{abc=1}{=}$$

$$=\sqrt{\frac{3}{2}(a+b+c+ab+bc+ca)}\leq\sqrt{\frac{3}{2}\left(a+b+c+\frac{(a+b+c)^2}{3}\right)}=$$

$$=\sqrt{\frac{3}{2}\left(x+\frac{x^2}{3}\right)}=\sqrt{\frac{x}{2}(x+3)}\ so, R.H.S\leq\sqrt{\frac{x}{2}(x+3)}$$

$$We\ need\ to\ show:\ \frac{3x^2}{3+2x}\geq\sqrt{\frac{x}{2}(x+3)}\ \ or\ \ \left(\frac{3x^2}{3+2x}\right)^2\geq\left(\sqrt{\frac{x}{2}(x+3)}\right)^2$$

$$18x^3\geq(3+2x)^2(3+x)\ or\ 14x^3-24x^2-45x-27\geq 0$$

$$(x-3)(14x^2+18x+9)\geq 0\ True\ as\ x\geq 3$$

Equality holds for a=b=c=1

2245. *Let a,b,c positive real numbers with $abc=1$. Prove that:*

$$(a^2+b+c)(b^2+c+a)(c^2+a+b)\geq 3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$(a+b+c)\overset{AM-GM}{\geq}3\sqrt[3]{abc}\overset{abc=1}{=}3\ or, x\overset{a+b+c=x}{\geq}3\ (1)$$

$$(1+a+b)(1+b+c)(1+c+a)\overset{AM-GM}{\leq}\left(\frac{(1+a+b)+(1+b+c)+(1+c+a)}{3}\right)^3$$

$$=\left(\frac{3+2(a+b+c)}{3}\right)^3\overset{(1)}{\leq}\left(\frac{(a+b+c)+2(a+b+c)}{3}\right)^3=(a+b+c)^3\ (2)$$

$$(a+b+c)^2\geq 3(ab+bc+c)\ (3)$$

$$(a^2+b+c)(1+b+c)\overset{C-S}{\geq}(a+b+c)^2\ or, (a^2+b+c)\geq\frac{(a+b+c)^2}{(1+b+c)}$$

$$Similarly:\ (b^2+c+a)\geq\frac{(a+b+c)^2}{1+c+a}\ \&\ (c^2+a+b)\geq\frac{(a+b+c)^2}{1+a+b}$$

$$(a^2+b+c)(b^2+c+a)(c^2+a+b)\geq\frac{(a+b+c)^6}{(1+a+b)(1+b+c)(1+c+a)}\geq$$

$$\overset{(2)}{\geq} \frac{(a+b+c)^6}{(a+b+c)^3} = (a+b+ç)^3 =$$

$$= (a+b+c)(a+b+c)^2 \overset{(3)}{\geq} 3(a+b+c)(ab+bc+ca) =$$

$$\overset{abc=1}{=} 3(a+b+c)\frac{(ab+bc+ca)}{abc} = 3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Equality holds for a=b=c=1.

2246.

If a,b,c be positive real numbers then prove that:

$$\left(\sqrt{a^2+8bc}-a\right)\left(\sqrt{b^2+8ca}-b\right)\left(\sqrt{c^2+8ab}-c\right) \leq \frac{64a^2b^2c^2}{(a+\sqrt[3]{abc})(b+\sqrt[3]{abc})(c+\sqrt[3]{abc})}$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$\left(\sqrt{a^2+8bc}-a\right) = \frac{\left(\sqrt{a^2+8bc}+a\right)\left(\sqrt{a^2+8bc}-a\right)}{\left(\sqrt{a^2+8bc}-a\right)} = \frac{8bc}{\left(\sqrt{a^2+8bc}-a\right)}$$

$$\textit{Similarly: } \left(\sqrt{b^2+8ca}-b\right) = \frac{8ca}{\left(\sqrt{b^2+8ca}+b\right)}$$

$$\left(\sqrt{c^2+8ab}-c\right) = \frac{8ab}{\left(\sqrt{c^2+8ab}+c\right)}$$

$$\left(\sqrt{a^2+8bc}-a\right)\left(\sqrt{b^2+8ca}-b\right)\left(\sqrt{c^2+8ab}-c\right) =$$

$$= \frac{512a^2b^2c^2}{\left(\sqrt{a^2+8bc}+a\right)\left(\sqrt{b^2+8ca}+b\right)\left(\sqrt{c^2+8ab}+c\right)}$$

$$\textit{We need to show: }$$

$$\left(\sqrt{a^2+8bc}-a\right)\left(\sqrt{b^2+8ca}-b\right)\left(\sqrt{c^2+8ab}-c\right) \leq$$

$$\leq \frac{64a^2b^2c^2}{(a+\sqrt[3]{abc})(b+\sqrt[3]{abc})(c+\sqrt[3]{abc})}$$

$$\frac{512a^2b^2c^2}{\left(\sqrt{a^2+8bc}+a\right)\left(\sqrt{b^2+8ca}+b\right)\left(\sqrt{c^2+8ab}+c\right)} \leq$$

$$\leq \frac{64a^2b^2c^2}{(a+\sqrt[3]{abc})(b+\sqrt[3]{abc})(c+\sqrt[3]{abc})}$$

$$\left(\sqrt{a^2+8bc}+a\right)\left(\sqrt{b^2+8ca}+b\right)\left(\sqrt{c^2+8ab}+c\right)\geq$$
$$\geq 8\left(a+\sqrt[3]{abc}\right)\left(b+\sqrt[3]{abc}\right)\left(c+\sqrt[3]{abc}\right)(A)$$

$$\textbf{Lemma}:\forall m>0,\ \sqrt{m^2+\frac{8}{m}}+m\geq\frac{2(m+1)}{\sqrt{m}}$$

$$\textbf{Proof}:\ \sqrt{m^2+\frac{8}{m}}+m\geq\frac{2(m+1)}{\sqrt{m}}\ or\ \sqrt{m^3+8}+m^{\frac{3}{2}}\geq 2m+2$$
$$or\ \sqrt{t^6+8}+t^3\overset{t^2=x}{\geq}2t^2+2\ or\ \left(\sqrt{t^6+8}\right)^2\geq(2t^2+2-t^3)^2$$
$$4\left(t^5-t^4+t^3-2t^2+1\right)\geq 0\ or, 4(t-1)^2(t^3+t^2+2t+1)\geq 0\ True$$
$$Let\ x=\frac{a}{\sqrt[3]{abc}}, y=\frac{b}{\sqrt[3]{abc}}, z=\frac{c}{\sqrt[3]{abc}}\ then\ xyz=1$$
$$\left(\sqrt{a^2+8bc}+a\right)=\sqrt[3]{abc}\left(\sqrt{\frac{a^2}{(abc)^{\frac{2}{3}}}+\frac{8bc}{(abc)^{\frac{2}{3}}}}+\frac{a}{\sqrt[3]{abc}}\right)=$$
$$=\sqrt[3]{abc}\left(\sqrt{\frac{a^2}{(abc)^{\frac{2}{3}}}+\frac{8}{\frac{a}{\sqrt[3]{abc}}}}+\frac{a}{\sqrt[3]{abc}}\right)=$$
$$=\sqrt[3]{abc}\left(\sqrt{x^2+\frac{8}{x}}+x\right)\overset{Lemma}{\geq}\sqrt[3]{abc}\ \frac{2(x+1)}{\sqrt{x}}$$

$$\left(\sqrt{a^2+8bc}+a\right)\left(\sqrt{b^2+8ca}+b\right)\left(\sqrt{c^2+8ab}+c\right)\geq$$

$$\geq\sqrt[3]{abc}\frac{2(x+1)}{\sqrt{x}}\sqrt[3]{abc}\frac{2(y+1)}{\sqrt{y}}\sqrt[3]{abc}\frac{2(z+1)}{\sqrt{z}}=$$
$$=\frac{8\left(\sqrt[3]{abc}\right)^3(x+1)(y+1)(z+1)}{\sqrt{xyz}}=$$
$$\overset{xyz=1}{=}8\left(\sqrt[3]{abc}\right)^3(x+1)(y+1)(z+1)=$$

$$=8\left(\sqrt[3]{abc}\right)^3\left(\frac{a}{\sqrt[3]{abc}}+1\right)\left(\frac{b}{\sqrt[3]{abc}}+1\right)\left(\frac{c}{\sqrt[3]{abc}}+1\right)=$$

$$=8\left(a+\sqrt[3]{abc}\right)\left(b+\sqrt[3]{abc}\right)\left(c+\sqrt[3]{abc}\right), so\ (A) is\ true$$

Equality holds for a=b=c.

2247. If $a,b,c>0$ then:

$$\frac{x^3+y^3+z^3}{(x+y+z)(x^2+y^2+z^2)}\geq\frac{1}{3}+\frac{(x-y)^2+(y-z)^2+(z-x)^2}{6(x^2+y^2+z^2)}$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$Schur's\ inequality:\ a^3+b^3+c^3+3abc\geq a^2(b+c)+b^2(c+a)+c^2(a+b)$$
$$a^3+b^3+c^3+3abc=(a+b+c)(a^2+b^2+c^2)-(a^3+b^3+c^3)$$
$$or,2(a^3+b^3+c^3)+3abc\geq(a+b+c)(a^2+b^2+c^2)\ (A)$$

$$We\ need\ to\ show:$$
$$\frac{x^3+y^3+z^3}{(x+y+z)(x^2+y^2+z^2)}\geq\frac{1}{3}+\frac{(x-y)^2+(y-z)^2+(z-x)^2}{6(x^2+y^2+z^2)}$$
$$or\ \frac{x^3+y^3+z^3}{(x+y+z)(x^2+y^2+z^2)}-\frac{(x-y)^2+(y-z)^2+(z-x)^2}{6(x^2+y^2+z^2)}\geq\frac{1}{3}$$

$$L.H.S=\frac{x^3+y^3+z^3}{(x+y+z)(x^2+y^2+z^2)}-\frac{(x-y)^2+(y-z)^2+(z-x)^2}{6(x^2+y^2+z^2)}=$$

$$=\frac{x^3+y^3+z^3}{(x+y+z)(x^2+y^2+z^2)}-\frac{(x^2+y^2+z^2)-(xy+yz+zx)}{3(x^2+y^2+z^2)}=$$

$$=\frac{3(x^3+y^3+z^3)-(x+y+z)(x^2+y^2+z^2)+(x+y+z)(xy+yz+zx)}{3(x^2+y^2+z^2)(x+y+z)}=$$

$$=\frac{3(x^3+y^3+z^3)-(x^3+y^3+z^3)-x^2(y+z)-y^2(z+x)-z^2(x+y)+(x+y+z)(xy+yz+zx)}{3(x^2+y^2+z^2)(x+y+z)}$$

$$=\frac{2(x^3+y^3+z^3)-x^2(y+z)-y^2(z+x)-z^2(x+y)+3xyz+x^2(y+z)+z^2(x+y)+y^2(x+z)}{3(x^2+y^2+z^2)(x+y+z)}$$

$$=\frac{2(x^3+y^3+z^3)+3xyz}{3(x+y+z)(x^2+y^2+z^2)}\overset{By\ (A)}{\geq}\frac{1}{3}$$

Equality holds for x=y=z.

2248. *Let a,b,c be positive real numbers. Prove that:*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\le\frac{3}{4}+\frac{(a+b+c)(ab+bc+ca)}{(a+b+c)^3+9abc}+\frac{\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}}{a^2+b^2+c^2}$$

Proposed by Le Anh Tu-Vietnam

Solution by Jenish Rijal-Bardiya-Nepal

$$\sum_{cyc}\frac{a(a^2+b^2+c^2)}{b+c}-\sum_{cyc}\frac{a^3}{b+c}=\sum_{cyc}\frac{a(b^2+c^2)}{b+c}=\sum_{cyc}\frac{a[(b+c)^2-2bc]}{b+c}=\sum_{cyc}\left(a(b+c)-\frac{2abc}{b+c}\right)=$$

$$=2(ab+bc+ca)-2abc\sum_{cyc}\frac{1}{b+c}\overset{Cauchy}{\le}$$

$$\le 2(ab+bc+ca)-\frac{9abc}{a+b+c}\quad\left\{as\ \sum_{cyc}\frac{1}{b+c}\overset{Bergstrom}{\ge}\frac{9}{2(a+b+c)}\right\}$$

$$\therefore\sum_{cyc}\frac{a}{b+c}-\frac{\sum\frac{a^3}{b+c}}{\sum a^2}\le\frac{2(ab+bc+ca)-\frac{9abc}{a+b+c}}{a^2+b^2+c^2}=\frac{2(a+b+c)(ab+bc+ca)-9abc}{(a+b+c)(a^2+b^2+c^2)}$$

Let $p:=a+b+c, q=ab+bc+ca$ and $r=abc$.

Also assume $S=p^3-4pq+9r$ which is $\overset{Schur}{\ge}0$. $\therefore S\ge 0$

So, we need to show that: $\frac{2pq-9r}{p(p^2-2q)}\le\frac{3}{4}+\frac{(a+b+c)(ab+bc+ca)}{(a+b+c)^3+9abc}=\frac{3}{4}+\frac{pq}{p^3+9r}$

We have:

$$\frac{2pq-9r}{p(p^2-2q)}=1-\frac{p^3-4pq+9r}{p(p^2-2q)}=1-\frac{S}{p(p^2-2q)}\ \text{and}\ \frac{3}{4}+\frac{pq}{p^3+9r}=$$

$$=1-\frac{p^3-4pq+9r}{4(p^3+9r)}=1-\frac{S}{4(p^3+9r)}$$

Thus, it suffices to prove that: $1-\frac{S}{p(p^2-2q)}\le 1-\frac{S}{4(p^3+9r)}$

$\Leftrightarrow S\cdot 4(p^3+9r)-S\cdot p(p^2-2q)\ge 0\ \Leftrightarrow S(3p^3+2pq+36r)\ge 0$

Which is true as $S\ge 0$. Equality holds if and only if $a=b=c$

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2249. Let a, b, c be positive real numbers. Prove the following inequalities:

$$\sum_{cyc}\sqrt{a^2+ab+b^2} \geq \sqrt{4\sum_{cyc}a^2+5\sum_{cyc}ab}$$

$$\sum_{cyc}\sqrt{a^2+ab+b^2} \leq \sqrt{5\sum_{cyc}a^2+4\sum_{cyc}ab}$$

Proposed by Le Anh Tu-Vietnam

Solution by Nawar Alasadi-Iraq

Proof of (1):

Let $L=\sum_{cyc}\sqrt{a^2+ab+b^2}$.

$$L^2 = \sum_{cyc}(a^2+ab+b^2)+2\sum_{cyc}\sqrt{a^2+ab+b^2}\sqrt{b^2+bc+c^2}$$

By the Cauchy-Schwarz inequality:

$$\begin{aligned}\sqrt{a^2+ab+b^2}\sqrt{b^2+bc+c^2} &= \sqrt{\left(b+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2}\sqrt{\left(b+\frac{c}{2}\right)^2+\left(\frac{\sqrt{3}c}{2}\right)^2}\\ &\geq \left(b+\frac{a}{2}\right)\left(b+\frac{c}{2}\right)+\frac{3ac}{4}\\ &= b^2+\frac{1}{2}ab+\frac{1}{2}bc+ac\end{aligned}$$

$$2\sum_{cyc}\sqrt{a^2+ab+b^2}\sqrt{b^2+bc+c^2} \geq 2\sum_{cyc}\left(b^2+\frac{1}{2}ab+\frac{1}{2}bc+ac\right)=2\sum_{cyc}a^2+4\sum_{cyc}ab$$

$$\begin{aligned}L^2 &\geq \left(2\sum_{cyc}a^2+\sum_{cyc}ab\right)+\left(2\sum_{cyc}a^2+4\sum_{cyc}ab\right)\\ &= 4\sum_{cyc}a^2+5\sum_{cyc}ab \Rightarrow (1)\end{aligned}$$

Proof of (2):

By the Cauchy-Schwarz inequality:

$$L^2 = \left(\sum_{cyc}\sqrt{a+b}\sqrt{\frac{a^2+ab+b^2}{a+b}}\right)^2 \le \left(\sum_{cyc}(a+b)\right)\left(\sum_{cyc}\frac{a^2+ab+b^2}{a+b}\right)$$

$$L^2 \le 2(a+b+c)\sum_{cyc}\left(a+b-\frac{ab}{a+b}\right) = 4(a+b+c)^2 - 2(a+b+c)\sum_{cyc}\frac{ab}{a+b}$$

We need to prove:

$$4(a+b+c)^2 - 2(a+b+c)\sum_{cyc}\frac{ab}{a+b} \le 5\sum_{cyc}a^2 + 4\sum_{cyc}ab$$

$$\sum_{cyc}a^2 - 4\sum_{cyc}ab + 2(a+b+c)\sum_{cyc}\frac{ab}{a+b} \ge 0$$

$$\sum_{cyc}a^2 - 2\sum_{cyc}ab + 2abc\sum_{cyc}\frac{1}{a+b} \ge 0$$

Let $p = a+b+c, q = ab+bc+ca, r = abc$. The inequality becomes:

$$p^2 - 4q + 2r\left(\frac{p^2+q}{pq-r}\right) \ge 0$$

$$(p^2-4q)(pq-r) + 2r(p^2+q) \ge 0 \quad (\text{since } pq-r>0)$$

$$p^3q - 4pq^2 + p^2r + 6qr \ge 0$$

Using Schur's inequality of the third degree ($p^3 - 4pq + 9r \ge 0$) and the identity $p^2 \ge 3q$:

$$p^3q - 4pq^2 + p^2r + 6qr \ge p^3q - 4pq^2 + 3qr + 6qr = q(p^3 - 4pq + 9r) \ge 0 \Rightarrow (2)$$

2250. If $a, b, c > 0$ then:

$$\prod_{cyc}\left(\sqrt{a^2+3b^2} - a\right) \le \frac{2abc\sqrt[3]{abc}}{a+b+c-\sqrt[3]{abc}}$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$Let\ x = \sqrt[3]{abc}\ \&\ A = \frac{a}{x}, B = \frac{b}{x}, C = \frac{c}{x}\ then\ ABC = 1$$

$$s = a+b+c = Ax + Bx + Cx = x(A+B+C) = xP,$$

$$(P = A+B+C), Q = AB+BC+CA$$

$$Q^2 \ge 3ABC(A+B+C) = 3P\ \&\ P \ge 3\sqrt[3]{ABC} = 3\ \bullet$$

$$a^2+3b^2 = \frac{a^2}{1} + \frac{b^2}{1} + \frac{b^2}{1} + \frac{b^2}{1} \overset{Radon}{\ge} \frac{(a+b+b+b)^2}{4} = \frac{(a+3b)^2}{4}\ (1)$$

$$\sqrt{a^2+3b^2}-a=\frac{\left(\sqrt{a^2+3b^2}-a\right)\left(\sqrt{a^2+3b^2}+a\right)}{\sqrt{a^2+3b^2}+a}=$$

$$=\frac{3b^2}{\sqrt{a^2+3b^2}+a}\overset{(1)}{\leq}\frac{3b^2}{\frac{a+3b}{2}+a}=\frac{2b^2}{a+b}$$

$$L.H.S\leq\frac{8a^2b^2c^2}{(a+b)(b+c)(c+a)}\ (A),$$

$$\frac{\prod(a+b)}{x^3}=(A+B)(B+C)(C+A)=\sum A\sum AB-ABC=PQ-1$$

$$\frac{4x^2(s-x)}{x^3}=4x^2.\frac{x(P-1)}{x^3}=4(P-1)$$

$$We\ need\ to\ show:$$

$$\frac{8a^2b^2c^2}{(a+b)(b+c)(c+a)}\leq\frac{2abc\sqrt[3]{abc}}{a+b+c-\sqrt[3]{abc}}=\frac{2abcx}{s-x}$$

$$or,\frac{\prod(a+b)}{4abc}\geq\frac{s-x}{x}=\frac{4x^2(s-x)}{x^3}\ or,\frac{\prod(a+b)}{x^3}\geq\frac{4x^2(s-x)}{x^3}\ or,$$

$$PQ-1\geq4(P-1)$$

$$or,P\sqrt{3P}\overset{Q\geq3P}{\geq}4P-3\ or,3P^3-16P^2+24P-9\geq0\ or,$$

$$(P-3)(3P^2-7P+3)\geq0$$

$$or,(P-3)(P(3P-7)+3)\geq0\ true\ as\ P\geq3$$

Equality holds for a=b=c

2251. *If $a,b,c>0$ then prove that:*

$$\frac{a+b+c}{\sqrt[3]{abc}}\geq2+\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$Let\ x=\frac{a^2+b^2+c^2}{\sqrt[3]{a^2b^2c^2}}\overset{AM-GM}{\geq}\frac{3\sqrt[3]{a^2b^2c^2}}{\sqrt[3]{a^2b^2c^2}}=3$$

$$y=\frac{ab+bc+ca}{\sqrt[3]{a^2b^2c^2}}\overset{AM-GM}{\geq}\frac{3\sqrt[3]{a^2b^2c^2}}{\sqrt[3]{a^2b^2c^2}}=3.and\ \ z=\frac{a^2+b^2+c^2}{ab+bc+ca}\geq1\ clearly,\frac{x}{y}=z$$

$$We\ need\ to\ show:$$

$$\frac{a+b+c}{\sqrt[3]{abc}}\geq2+\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}\ or\ \sqrt{\frac{(a+b+c)^2}{\sqrt[3]{a^2b^2c^2}}}\geq2+\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}$$

$$\sqrt{\frac{a^2+b^2+c^2+2(ab+bc+ca)}{\sqrt[3]{a^2b^2c^2}}} \geq 2+\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}$$

$$\sqrt{\frac{a^2+b^2+c^2}{\sqrt[2]{a^2b^2c^2}}+2\ \frac{ab+bc+ca}{\sqrt[3]{a^2b^2c^2}}} \geq 2+\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}$$

$$\sqrt{x+2y} \geq 2+\sqrt{z}\ or\ \sqrt{x+2y} \geq 2+\sqrt{\frac{x}{y}}\ or\ \sqrt{y\left(\frac{x}{y}+2\right)} \geq 2+\sqrt{\frac{x}{y}}$$

$$\sqrt{3\left(\frac{x}{y}+2\right)} \overset{y\geq 3}{\geq} 2+\sqrt{\frac{x}{y}}\ or\ 3\left(\frac{x}{y}+2\right) \geq 4+\frac{x}{y}+4\sqrt{\frac{x}{y}}\ or\ 2\left(\frac{x}{y}-2\sqrt{\frac{x}{y}}+1\right) \geq 0$$

$$or\ \left(\sqrt{\frac{x}{y}}-1\right)^2 \geq 0\ True.\ \text{Equality holds for a=b=c}$$

2252. If $x,y,z>0, \sum_{cyc} x=1$ and $\lambda \geq 1$ then :

$$\sum_{cyc} \frac{x^2}{\lambda-x^2} \geq \frac{3}{9\lambda-1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\lambda-x^2 \geq 1-x^2=(1+x)(y+z)>0\ (\because x,y,z>0)$$

$$\therefore \lambda-x^2>0 \text{ and analogs } \therefore \sum_{cyc}\frac{x^2}{\lambda-x^2} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc}x\right)^2}{3\lambda-\sum_{cyc}x^2} \overset{?}{\geq} \frac{3}{9\lambda-1}$$

$$\Leftrightarrow 9\lambda\left(\left(\sum_{cyc}x\right)^2-1\right)+3\sum_{cyc}x^2-\left(\sum_{cyc}x\right)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow 2\left(\sum_{cyc}x^2-\sum_{cyc}xy\right) \overset{?}{\geq} 0\ \left(\because \sum_{cyc}x=1\right) \rightarrow \text{true} \therefore \sum_{cyc}\frac{x^2}{\lambda-x^2} \geq \frac{3}{9\lambda-1}$$

$$\forall x,y,z>0 \mid \sum_{cyc}x=1 \text{ and } \lambda \geq 1, ''='' \text{ iff } x=y=z=\frac{1}{3}\ (\text{QED})$$

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2253. *Let $p,q,r>0$ such that $pqr\geq 1$. Prove that:*

$$\frac{p^n\sqrt{p^nq^n+r^np^n+2q^nr^n}}{q^n+r^n}+\frac{q^n\sqrt{q^nr^n+p^nq^n+2r^np^n}}{r^n+p^n}+\frac{r^n\sqrt{r^np^n+q^nr^n+2p^nq^n}}{p^n+q^n}\geq$$

$$\geq 3\left(\frac{1}{p^5+q^2+r^2}+\frac{1}{q^5+r^2+p^2}+\frac{1}{r^5+p^2+q^2}\right);\ \textit{where } n \textit{ is a real number} \geq 1.$$

Proposed by Michael Brooks-USA

Solution by Jenish Rijal-Bardiya-Nepal

Let $x=p^n, y=q^n,$ and $z=r^n$. *Then the given inequality becomes,*

$$\sum_{cyc}\frac{x\sqrt{xy+zx+2yz}}{y+z}\geq 3\sum_{cyc}\frac{1}{p^5+q^2+r^2}$$

We have, $\frac{x\sqrt{xy+zx+2yz}}{y+z}+\frac{x\sqrt{xy+zx+2yz}}{y+z}+\frac{x(y+z)^2}{xy+zx+2yz}\overset{AM-GM}{\geq} 3x$

$$\therefore 2\sum_{cyc}\frac{x\sqrt{xy+zx+2yz}}{y+z}+\sum_{cyc}\frac{x(y+z)^2}{xy+zx+2yz}\geq 3(x+y+z)$$

$$\textit{Also,}\ \sum_{cyc}\frac{x(y+z)^2}{xy+zx+2yz}=\sum_{cyc}\frac{\left(\sqrt{x}y+z\sqrt{x}\right)^2}{(xy+yz)+(yz+zx)}\overset{Bergstrom}{\leq}$$

$$\leq\sum_{cyc}\frac{xy^2}{(xy+yz)}+\sum_{cyc}\frac{z^2x}{(yz+zx)}=\sum_{cyc}\frac{xy}{x+z}+\sum_{cyc}\frac{zx}{y+x}=x+y+z$$

$$\textit{So,}\quad \sum_{cyc}\frac{x\sqrt{xy+zx+2yz}}{y+z}\geq\frac{1}{2}\left[3(x+y+z)-\underbrace{\sum_{cyc}\frac{x(y+z)^2}{xy+zx+2yz}}_{\leq x+y+z}\right]\geq x+y+z$$

$$\therefore\sum_{cyc}\frac{p^n\sqrt{p^nq^n+r^np^n+2q^nr^n}}{q^n+r^n}\geq p^n+q^n+r^n\overset{AM-GM}{\geq}3\sqrt[3]{(pqr)^n}\overset{pqr\geq 1}{\geq}3\sqrt[3]{1}=3$$

$$\textit{Thus,}\sum_{cyc}\frac{p^n\sqrt{p^nq^n+r^np^n+2q^nr^n}}{q^n+r^n}\geq 3\cdot 1\overset{\sum p^2\geq 3}{\geq}3\cdot\frac{3}{p^2+q^2+r^2}=\frac{9}{p^2+q^2+r^2}$$

$$\textit{Hence, it suffices to prove that:}\ \sum_{cyc}\frac{1}{p^5+q^2+r^2}\leq\frac{3}{p^2+q^2+r^2}$$

$$Here, \left(p^5+q^2+r^2\right)\left(\frac{1}{p}+q^2+r^2\right) \overset{Cauchy}{\geq} \left(p^2+q^2+r^2\right)^2 \Leftrightarrow \frac{1}{p^5+q^2+r^2} \leq \frac{\left(\frac{1}{p}+q^2+r^2\right)}{(p^2+q^2+r^2)^2}$$

$$\therefore \sum_{cyc} \frac{1}{p^5+q^2+r^2} \leq \sum_{cyc} \frac{\left(\frac{1}{p}+q^2+r^2\right)}{(p^2+q^2+r^2)^2} = \frac{2\sum p^2+\sum \frac{1}{p}}{(\sum p^2)^2} \overset{\sum \frac{1}{p} \leq \sum pq}{\leq}$$

$$\leq \frac{2\sum p^2+\sum pq}{(\sum p^2)^2} \overset{\sum pq \leq \sum p^2}{\leq} \frac{2\sum p^2+\sum p^2}{(\sum p^2)^2} = \frac{3}{\sum p^2}$$

Equality holds if and only if $p=q=r=1$.

2254. If $a,b>0$ and $ab=1$ then:

$$\frac{a}{b^2+3a}+\frac{b}{a^2+3b} \leq \frac{1}{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{Denote: } \begin{cases} S=a+b \\ P=ab=1 \end{cases} \overset{AM-GM}{\geq} 2\sqrt{ab}=2\sqrt{1}=2$$

$$S \geq 2 \Rightarrow S-2 \geq 0 \qquad (1)$$

$$\frac{a}{b^2+3a}+\frac{b}{a^2+3b} \leq \frac{1}{2}$$

$$2a(a^2+2b)+2b(b^2+3a) \leq (b^2+3a)(a^2+3b)$$

$$2a^3+6ab+2b^3+6ab \leq (ab)^2+3b^3+3a^3+9ab$$

$$2(a^3+b^3)+6+6 \leq 1+3(b^3+a^3)+9$$

$$a^3+b^3+10 \geq 12 \Leftrightarrow a^3+b^3 \geq 2 \qquad \text{(to prove)}$$

$$a^3+b^3 \geq 2\sqrt{(ab)^3}=2\sqrt{1^3}=2\cdot 1=2$$

Equality holds for $a=b=1$.

2255. If $a,b>0; ab=1$ then:

$$\frac{a}{b^3+3a}+\frac{b}{a^3+3b} \leq \frac{1}{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{b^3+3a}+\frac{b}{a^3+3b}\le\frac{1}{2}$$

$$2a(a^3+3b)+2b(b^3+3a)\le(b^3+3a)(a^3+3b)$$

$$2a^4+6ab+2b^4+6ab\le(ab)^3+3b^4+3a^4+9ab$$

$$2(a^4+b^4)+6+6\le 1+3(a^4+b^4)+9$$

$$a^4+b^4\ge 12-10=2$$

$$a^4+b^4\ge 2 \quad \text{(to prove)}$$

$$a^4+b^4\overset{AM-GM}{\ge}2\sqrt{(ab)^4}=2\sqrt{1^4}=2\cdot 1=2$$

Equality holds for $a=b=1$.

2256. If $a,b,c>0, abc=8$ and $0\le\lambda\le 1$ then :

$$\sum_{cyc}\frac{1}{a+\lambda}\ge\frac{3}{\lambda+2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{a+\lambda}\overset{?}{\ge}\frac{3}{\lambda+2}\Leftrightarrow\frac{3\lambda^2+2\lambda\sum_{cyc}a+\sum_{cyc}ab}{abc+\lambda\sum_{cyc}ab+\lambda^2\sum_{cyc}a+\lambda^3}\overset{?}{\ge}\frac{3}{\lambda+2}$$

$$\Leftrightarrow\left(6-\sum_{cyc}a\right)\lambda^2+\left(4\sum_{cyc}a-2\sum_{cyc}ab\right)\lambda+2\sum_{cyc}ab-3abc\ \boxed{\overset{?}{\ge}}\ 0 \quad (*)$$

$$\text{Firstly, }\sum_{cyc}a\overset{AM-GM}{\ge}3.\sqrt[3]{abc}=3.\sqrt[3]{8}=6\Rightarrow\sum_{cyc}a\ge 6\to(1)\text{ and secondly,}$$

$$2\sum_{cyc}ab\overset{AM-GM}{\ge}6.\sqrt[3]{a^2b^2c^2}=6.\sqrt[3]{\frac{a^3b^3c^3}{abc}}\overset{abc=8}{=}6.\sqrt[3]{\frac{a^3b^3c^3}{8}}=3abc$$

$$\Rightarrow 2\sum_{cyc}ab\ge 3abc\to(2)$$

Case 1 $4\sum_{cyc}a-2\sum_{cyc}ab\ge 0$ and $\because 0\le\lambda\le 1\Rightarrow\lambda\ge\lambda^2$ and $1\ge\lambda^2$, then, via (2),

$$\text{LHS} \geq \left(6-\sum_{cyc} a\right)\lambda^2+\left(4\sum_{cyc} a-2\sum_{cyc} ab\right)\lambda^2+\left(2\sum_{cyc} ab-3abc\right)\lambda^2$$

$$=\lambda^2\left(3\sum_{cyc} a+6-3abc\right)\overset{AM-GM}{\geq}\lambda^2\left(9.\sqrt[3]{abc}+6-3abc\right)\left(\because \lambda^2\geq 0\right)\overset{abc=8}{=}0$$

$$\Rightarrow (*) \text{ is true}$$

Case 2 $4\sum_{cyc} a-2\sum_{cyc} ab<0$ and then, since $\lambda,\lambda^2\leq 1$, then via ①, LHS of $(*)\geq$

$$6-\sum_{cyc} a+4\sum_{cyc} a-2\sum_{cyc} ab+2\sum_{cyc} ab-3abc=3\sum_{cyc} a+6-3abc\overset{AM-GM}{\geq}$$

$$9.\sqrt[3]{abc}+6-3abc\overset{abc=8}{=}0\Rightarrow (*) \text{ is true and so, combining both cases,}$$

$$\sum_{cyc}\frac{1}{a+\lambda}\geq\frac{3}{\lambda+2}\ \forall a,b,c>0\,|\,abc=8 \text{ and } 0\leq\lambda\leq 1,$$

$$''='' \text{ iff } a=b=c=2 \text{ (QED)}$$

2257. *If $a,b,c>0$ & $abc=1$ then prove that :*

$$\sum_{cyc}\sqrt{\frac{a^4+b^4+a^2b^2+1}{2(a^2+b^2)}}+\sqrt[4]{\frac{(a^3+b^3+c^3)^5}{9(a+b+c)(a^2+b^2+c^2)^2}}+\prod_{cyc}\frac{a^2+b^2+2}{a+b+2}\geq 5$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Tapas Das-India

$$2a^4+a^2b^2+1\overset{AM-GM}{\geq}4a^{\frac{5}{2}}b^{\frac{1}{2}}=4a^2\sqrt{ab}\ \&\ 2b^4+a^2b^2+1\overset{AM-GM}{\geq}4b^{\frac{5}{2}}a^{\frac{1}{2}}=4b^2\sqrt{ab}$$

$$(2a^4+a^2b^2+1)+(2b^4+a^2b^2+1)\geq 4\sqrt{ab}(a^2+b^2)$$

$$\frac{a^4+b^4+a^2b^2+1}{2(a^2+b^2)}\geq\sqrt{ab}$$

$$(a^3+b^3+c^3)\overset{Chebyshev}{\geq}\frac{1}{3}(a^2+b^2+c^2)(a+b+c)$$

$$\sqrt[4]{\frac{(a^3+b^3+c^3)^5}{9(a+b+c)(a^2+b^2+c^2)^2}}\geq\sqrt[4]{\frac{(a^2+b^2+c^2)^3(a+b+c)^4}{9\times 3^5}}\overset{AM-GM}{\geq}$$

$$\geq\sqrt[4]{\frac{\left(3(abc)^{\frac{2}{3}}\right)^3.\left(3(abc)^{\frac{1}{3}}\right)^4}{9\times 3^5}}\overset{abc=1}{=}1$$

$$a^2+b^2+2=\frac{a^2}{1}+\frac{b^2}{1}+\frac{1^2}{1}+\frac{1^2}{1}\overset{Radon}{\geq}\frac{1}{4}(a+b+2)^2$$

$$\prod_{cyc}\frac{a^2+b^2+2}{a+b+2}\geq\frac{1}{64}\prod_{cyc}(a+b+2)=\frac{1}{64}\prod_{cyc}(a+b+1+1)\overset{AM-GM}{\geq}$$

$$\geq\frac{1}{64}\prod_{cyc}4(a.b.1.1)^{\frac{1}{4}}=\frac{1}{64}(a^2b^2c^2)^{\frac{1}{4}}\times 64\overset{abc=1}{=}1$$

$$L.H.S\geq 3+1+1=5$$

Equality holds for a=b=c=1.

2258. *Let a,b,c be positive real numbers with $a+b+c=1$ Prove that:*

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{a+c}}+\frac{c}{\sqrt{b+a}}\geq\sqrt{\frac{3}{2}}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{a+c}}+\frac{c}{\sqrt{b+a}}=\frac{a^{\frac{3}{2}}}{\sqrt{ab+ac}}+\frac{b^{\frac{3}{2}}}{\sqrt{cb+ab}}+\frac{c^{\frac{3}{2}}}{\sqrt{bc+ac}}\overset{Radon}{\geq}$$

$$\geq\frac{(\sum_{cyc}a)^{\frac{3}{2}}}{(2\sum_{cyc}ab)^{\frac{1}{2}}}=\frac{1}{(2\sum_{cyc}ab)^{\frac{1}{2}}}\geq\frac{\sqrt{3}}{\sqrt{2}\sum_{cyc}a}=\sqrt{\frac{3}{2}}$$

$$Equality\ holds\ if:\ a=b=c=\frac{1}{3}$$

2259. *Let a,b,c be positive real numbers with $a+b+c\geq 3$. Prove that:*

$$\sum_{cyc}\frac{a^8}{a^2+nbc}+\frac{\sum_{cyc}a^2}{n+1}\geq\frac{6}{n+1},\qquad n>0$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{a^8}{a^2+nbc}+\frac{\sum_{cyc}a^2}{n+1}=\sum_{cyc}\frac{(a^2)^4}{a^2+nbc}+\frac{\sum_{cyc}a^2}{n+1}\overset{Bergstrom}{\geq}$$

$$\geq \frac{\left(\sum_{cyc} a^2\right)^4}{9\left(\sum_{cyc} a^2 + n\sum_{cyc} bc\right)} + \frac{\sum_{cyc} a^2}{n+1} \geq \frac{\left(\sum_{cyc} a^2\right)^4}{9\left(\sum_{cyc} bc + n\sum_{cyc} bc\right)} + \frac{\sum_{cyc} a^2}{n+1} =$$

$$= \frac{\left(\sum_{cyc} a^2\right)^3}{9(n+1)} + \frac{\sum_{cyc} a^2}{n+1} \geq \frac{\left(\frac{\left(\sum_{cyc} a\right)^2}{3}\right)^3}{9(n+1)} + \frac{\left(\sum_{cyc} a\right)^2}{3(n+1)} =$$

$$= \frac{\left(\sum_{cyc} a\right)^6}{27.9(n+1)} + \frac{\left(\sum_{cyc} a\right)^2}{3(n+1)} \geq \frac{3^6}{27.9(n+1)} + \frac{9}{3(n+1)} = \frac{3}{n+1} + \frac{3}{n+1} = \frac{6}{n+1}$$

$$Equality\ holds\ \ if:\ \ a = b = c = 1$$

2260. If $a, b, c > 0$ then:

$$\frac{a+b+c}{\sqrt[3]{abc}} \geq 1 + \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{abc}}$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$Let\ x = (a+b+c), y = \sqrt[3]{abc}\ then\ by\ AM - GM\ x \geq 3y\ (1)$$

$$We\ will\ show:$$

$$\left((a+b+c) - \sqrt[3]{abc}\right)^3 \geq (a+b)(b+c)(c+a)$$

$$Proof: (a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc \leq$$

$$\leq (a+b+c)\frac{(a+b+c)^2}{3} - abc \overset{\substack{a+b+c=x \\ (abc)^{\frac{1}{3}}=y}}{=} \frac{x^3}{3} - y^3 = \frac{x^3 - 3y^3}{3}$$

$$\left((a+b+c) - \sqrt[3]{abc}\right)^3 = (x-y)^3 = x^3 - y^3 - 3xy(x-y)$$

$$we\ need\ to\ show:\ x^3 - y^3 - 3xy(x-y) \geq \frac{x^3 - 3y^3}{3}\ or, 2x^3 \geq 9xy(x-y)$$

$$or, 2x^2 - 9xy + 9y^2 \geq 0\ or, (x-3y)(2x-3y) \geq 0\ True\ by\ (1)$$

$$\frac{a+b+c}{\sqrt[3]{abc}} = 1 + \left(\frac{a+b+c}{\sqrt[3]{abc}} - 1\right) = 1 + \frac{a+b+c-\sqrt[3]{abc}}{\sqrt[3]{abc}} =$$

$$= 1 + \sqrt[3]{\frac{\left((a+b+c) - \sqrt[3]{abc}\right)^3}{abc}} \geq 1 + \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{abc}}$$

Equality holds for a=b=c

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2261. If $a,b,c>0$ then:

$$\sum_{cyc}\frac{xy(x^2-xy+y^2)}{x^2+xy+y^2}\le\frac{2}{3}\sum_{cyc}\frac{x^4}{x^2+yz}$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$Lemma:\forall a,b>0,\frac{ab(a^2-ab+b^2)}{a^2+ab+b^2}\overset{AM-GM}{\le}\frac{ab(a^2-ab+b^2)}{3ab}=\frac{1}{3}(a^2-ab+b^2)$$

$$L.H.S=\sum_{cyc}\frac{xy(x^2-xy+y^2)}{x^2+xy+y^2}\overset{lemma}{\le}\frac{1}{3}\sum(x^2-xy+y^2)=\frac{2}{3}\sum x^2-\frac{1}{3}\sum xy$$

$$R.H.S=\frac{2}{3}\sum_{cyc}\frac{x^4}{x^2+yz}=\frac{2}{3}\sum\left(x^2-\frac{x^2yz}{x^2+yz}\right)\overset{AM-GM}{\le}\frac{2}{3}\sum x^2-\frac{2}{3}\sum\frac{x^2yz}{2x\sqrt{yz}}=$$

$$=\frac{2}{3}\sum x^2-\frac{1}{3}\sum x\sqrt{yz}\overset{AM-GM}{\le}\frac{2}{3}\sum x^2-\frac{1}{3}\sum\frac{x(y+z)}{2}=\frac{2}{3}\sum x^2-\frac{1}{3}\sum xy$$

$$so\ L.H.S\le R.H.S.$$

$$Equality\ holds\ for\ x=y=z$$

2262. If $x,y,z>0$ and $\lambda\le 3$ then :

$$\sum_{cyc}\frac{1}{xy}+\frac{\lambda}{\sum_{cyc}x^2}\ge\frac{\lambda+9}{\sum_{cyc}xy}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{xy}+\frac{\lambda}{\sum_{cyc}x^2}\overset{?}{\ge}\frac{\lambda+9}{\sum_{cyc}xy}\Leftrightarrow\frac{\sum_{cyc}x}{xyz}+\frac{\lambda}{\sum_{cyc}x^2}\overset{?}{\ge}\frac{\lambda+9}{\sum_{cyc}xy}$$

$$\Leftrightarrow\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-9xyz\right)\left(\sum_{cyc}x^2\right)\overset{?}{\ge}\lambda.xyz\left(\sum_{cyc}x^2-\sum_{cyc}xy\right)\ and\ \because$$

$$\sum_{cyc}x^2-\sum_{cyc}xy\ge 0\ and\ \lambda\le 3\ \therefore\ it\ suffices\ to\ prove:$$

$$\left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 9xyz\right)\left(\sum_{cyc} x^2\right) \overset{?}{\underset{(*)}{\geq}} 3.xyz\left(\sum_{cyc} x^2 - \sum_{cyc} xy\right)$$

Assigning $y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z>0, b+c-a=2x>0$ and $c+a-b=2y>0 \Rightarrow a+b>c, b+c>a, c+a>b$

$\Rightarrow a, b, c$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} x = \sum_{cyc} a = 2s \Rightarrow \sum_{cyc} x = s$

$$\Rightarrow x = s-a, y = s-b, z = s-c \Rightarrow xyz = r^2 s, \sum_{cyc} xy = 4Rr + r^2,$$

$$\sum_{cyc} x^2 = s^2 - 8Rr - 2r^2 \text{ and so, } (*) \Leftrightarrow$$

$$(s(4Rr+r^2) - 9r^2 s)(s^2 - 8Rr - 2r^2) \overset{?}{\geq} 3r^2 s(s^2 - 8Rr - 2r^2 - 4Rr - r^2)$$

$$\Leftrightarrow (4R-11r)s^2 \overset{?}{\underset{(**)}{\geq}} r(32R^2 - 92Rr - 25r^2)$$

Case 1 $4R - 11r \geq 0$ and then : LHS of $(**) \overset{\text{Gerretsen}}{\geq} (4R-11r)(16Rr - 5r^2) \overset{?}{\geq}$ RHS of $(**) \Leftrightarrow 8r(R-2r)(4R-5r) \overset{?}{\geq} 0 \rightarrow$ true $\because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (**)$ is true

Case 2 $4R - 11r < 0$; then : LHS of $(**) \overset{\text{Gerretsen}}{\geq} (4R-11r)(4R^2 + 4Rr + 3r^2) \overset{?}{\geq}$ RHS of $(**) \Leftrightarrow 4t^3 - 15t^2 + 15t - 2 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(4t^2 - 7t + 1) \overset{?}{\geq} 0$

$\rightarrow$ true $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**)$ is true $\therefore$ combining both cases, $(**) \Rightarrow (*)$ is true

$\forall \Delta XYZ_{s,R,r}$ and so, $\sum_{cyc} \frac{1}{xy} + \frac{\lambda}{\sum_{cyc} x^2} \geq \frac{\lambda+9}{\sum_{cyc} xy}$ $\forall x, y, z > 0$ and $\lambda \leq 3$,

$''='' \text{ iff } x = y = z$ (QED)

2263. If $a, b, c > 0, abc = 8$ and $0 \leq \lambda \leq 1$ then :

$$\sum_{cyc} \frac{1}{a+\lambda} \geq \frac{3}{\lambda+2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{1}{a+\lambda} \overset{?}{\geq} \frac{3}{\lambda+2} \Leftrightarrow \frac{3\lambda^2 + 2\lambda\sum_{cyc} a + \sum_{cyc} ab}{abc + \lambda\sum_{cyc} ab + \lambda^2\sum_{cyc} a + \lambda^3} \overset{?}{\geq} \frac{3}{\lambda+2}$$

$$\Leftrightarrow \left(6-\sum_{cyc} a\right)\lambda^2+\left(4\sum_{cyc} a-2\sum_{cyc} ab\right)\lambda+2\sum_{cyc} ab-3abc \overset{?}{\geq} 0 \quad (*)$$

$$\text{Firstly, } \sum_{cyc} a \overset{AM-GM}{\geq} 3.\sqrt[3]{abc}=3.\sqrt[3]{8}=6\Rightarrow \sum_{cyc} a\geq 6\rightarrow (1) \text{ and secondly,}$$

$$2\sum_{cyc} ab \overset{AM-GM}{\geq} 6.\sqrt[3]{a^2b^2c^2}=6.\sqrt[3]{\frac{a^3b^3c^3}{abc}} \overset{abc=8}{=} 6.\sqrt[3]{\frac{a^3b^3c^3}{8}}=3abc$$

$$\Rightarrow 2\sum_{cyc} ab\geq 3abc\rightarrow (2)$$

Case 1 $4\sum_{cyc} a-2\sum_{cyc} ab\geq 0$ and $\because 0\leq\lambda\leq 1\Rightarrow\lambda\geq\lambda^2$ and $1\geq\lambda^2$, then, via (2),

$$LHS\geq \left(6-\sum_{cyc} a\right)\lambda^2+\left(4\sum_{cyc} a-2\sum_{cyc} ab\right)\lambda^2+\left(2\sum_{cyc} ab-3abc\right)\lambda^2$$

$$=\lambda^2\left(3\sum_{cyc} a+6-3abc\right) \overset{AM-GM}{\geq} \lambda^2\left(9.\sqrt[3]{abc}+6-3abc\right)\ (\because \lambda^2\geq 0) \overset{abc=8}{=} 0$$

$$\Rightarrow (*) \text{ is true}$$

Case 2 $4\sum_{cyc} a-2\sum_{cyc} ab<0$ and then, since $\lambda,\lambda^2\leq 1$, then via (1), LHS of $(*)\geq$

$$6-\sum_{cyc} a+4\sum_{cyc} a-2\sum_{cyc} ab+2\sum_{cyc} ab-3abc=3\sum_{cyc} a+6-3abc \overset{AM-GM}{\geq}$$

$$9.\sqrt[3]{abc}+6-3abc \overset{abc=8}{=} 0\Rightarrow (*) \text{ is true and so, combining both cases,}$$

$$\sum_{cyc}\frac{1}{a+\lambda}\geq\frac{3}{\lambda+2}\ \forall a,b,c>0\ \big|\ abc=8 \text{ and } 0\leq\lambda\leq 1,$$

$$''='' \text{ iff } a=b=c=2 \text{ (QED)}$$

2264.

Let a,b,c be positive real numbers. Prove the following inequalities:

$$\sqrt{a^2+ab+b^2}+\sqrt{b^2+bc+c^2}+\sqrt{c^2+ca+a^2}\geq\sqrt{4(a^2+b^2+c^2)+5(ab+bc+ca)}$$

$$\sqrt{a^2+ab+b^2}+\sqrt{b^2+bc+c^2}+\sqrt{c^2+ca+a^2}\leq\sqrt{5(a^2+b^2+c^2)+4(ab+bc+ca)}$$

Proposed by Le Anh Tu-Vietnam

Solutions by Jenish Rijal-Nepal

Solution for problem 1:

$$\sum_{cyc}\sqrt{a^2+ab+b^2}\geq\sqrt{4\sum_{cyc}a^2+5\sum_{cyc}ab}$$

$$\overset{Squaring}{\Leftrightarrow}\sum_{cyc}(a^2+ab+b^2)+2\sum_{cyc}\left(\sqrt{a^2+ab+b^2}\cdot\sqrt{a^2+ac+c^2}\right)\overset{?}{\geq}$$

$$\geq 4\sum_{cyc}a^2+5\sum_{cyc}ab$$

$$\Leftrightarrow 2\sum_{cyc}\left(\sqrt{a^2+ab+b^2}\cdot\sqrt{a^2+ac+c^2}\right)\overset{?}{\geq}4\sum_{cyc}a^2+5\sum_{cyc}ab-\sum_{cyc}(a^2+ab+b^2)$$

Thus, it suffices to prove that: $\sum_{cyc}\left(\sqrt{a^2+ab+b^2}\cdot\sqrt{a^2+ac+c^2}\right)$

$$\geq\sum_{cyc}a^2+2\sum_{cyc}ab$$

Notice that: $a^2+ab+b^2=\frac{1}{2}[a^2+b^2+(a+b)^2]$ ***(and analogs)***

$$\sum_{cyc}\left(\sqrt{a^2+ab+b^2}\cdot\sqrt{a^2+ac+c^2}\right)=$$

$$=\sum_{cyc}\left(\sqrt{\frac{1}{2}[a^2+b^2+(a+b)^2]}\cdot\sqrt{\frac{1}{2}[a^2+c^2+(a+c)^2]}\right)=$$

$$=\sum_{cyc}\left(\frac{1}{2}\sqrt{a^2+b^2+(a+b)^2}\cdot\sqrt{a^2+c^2+(a+c)^2}\right)\overset{Cauchy}{\geq}$$

$$\geq\frac{1}{2}\sum_{cyc}[a(a)+bc+(a+b)(a+c)]=$$

$$=\frac{1}{2}\sum_{cyc}[2a^2+ab+2bc+ac]=\sum_{cyc}a^2+2\sum_{cyc}ab$$

Solution for problem 2:

$$\left(\sum_{cyc}\sqrt{a^2+ab+b^2}\right)^2=\left(\sum_{cyc}\sqrt{a+b}\cdot\sqrt{\frac{a^2+ab+b^2}{a+b}}\right)^2\overset{Cauchy}{\leq}\left(\sum_{cyc}(a+b)\right)\left(\sum_{cyc}\frac{a^2+ab+b^2}{a+b}\right)$$

Also, $$\sum_{cyc}\frac{a^2+ab+b^2}{a+b}=\sum_{cyc}(a+b)-\frac{ab}{a+b}=2(a+b+c)-\sum_{cyc}\frac{ab}{a+b}$$

$$\therefore\left(\sum_{cyc}\sqrt{a^2+ab+b^2}\right)^2\leq 2(a+b+c)\left[2(a+b+c)-\sum_{cyc}\frac{ab}{a+b}\right]=$$

$$=4(a+b+c)^2-\sum_{cyc}\frac{2ab(a+b+c)}{a+b}$$

$$\sum_{cyc}\frac{2ab(a+b+c)}{a+b}=\sum_{cyc}2ab\left(1+\frac{c}{a+b}\right)=2(ab+bc+ca)+\sum_{cyc}\frac{2abc}{a+b}=$$

$$=2(ab+bc+ca)+2abc\sum_{cyc}\frac{1}{a+b}$$

$$\therefore\left(\sum_{cyc}\sqrt{a^2+ab+b^2}\right)^2\leq 4(a+b+c)^2-2(ab+bc+ca)-2abc\sum_{cyc}\frac{1}{a+b}$$

By Titu's inequality, we have: $\sum_{cyc}\frac{1}{a+b}\geq\frac{9}{2(a+b+c)}$ ***. So,***

$$\left(\sum_{cyc}\sqrt{a^2+ab+b^2}\right)^2\leq 4(a+b+c)^2-2(ab+bc+ca)-\frac{9abc}{a+b+c}$$

It suffices to prove that:

$$4(a+b+c)^2-2(ab+bc+ca)-\frac{9abc}{a+b+c}\leq 5(a^2+b^2+c^2)+4(ab+bc+ca)$$

$$\Leftrightarrow a^2+b^2+c^2+\frac{9abc}{a+b+c}\geq 2(ab+bc+ca),$$

which is true as it is Schur's inequality of degree 3.

$Equality\ holds\ iff\ and\ only\ a=b=c.$

2265. *Let a,b,c,d be positive real numbers satisfying $a>b>c>d$. Find maximum value of*

$$P=\frac{(a-b)(a-c)(b-c)(b-d)(c-d)}{(a-d)^5}$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$Let\ a'=a-d, b'=b-d, c'=c-d\ \&\ a>b>c>d\Rightarrow a'>b'>c'>0$$

$$Let\ x=\frac{b'}{a'}, y=\frac{c'}{a'}\ , since\ a'>b'>c'>0\ then\ 1>x>y>0$$

$$using\ above\ substitution\ we\ get\ P=\frac{(a'-b')(a'-c')(b'-c')b'c'}{(a')^5}$$

$$now\ we\ take\ f(x,y)=P^{x=\frac{b'}{a'},y=\frac{c'}{a'}}=(1-x)(1-y)(x-y)xy$$

$$Let\ g(x,y)=\ln f(x,y)=\ln x+\ln y+\ln(x-y)+\ln(1-x)+\ln(1-y)$$

$$g_x=\frac{\delta g}{\delta x}=\frac{1}{x}-\frac{1}{1-x}+\frac{1}{x-y}=\frac{1}{x-y}+\frac{1-2x}{x(1-x)}, g_{xx}=\frac{\delta^2 g}{\delta x^2}$$
$$=-\frac{1}{x^2}-\frac{1}{(1-x)^2}-\frac{1}{(x-y)^2}$$

$$g_y=\frac{\delta g}{\delta y}=\frac{1}{y}-\frac{1}{1-y}-\frac{1}{x-y}=\frac{1-2y}{y(1-y)}-\frac{1}{x-y}, g_{yy}=\frac{\delta^2 g}{\delta y^2}$$
$$=-\frac{1}{y^2}-\frac{1}{(1-y)^2}-\frac{1}{(x-y)^2}$$

$$g_{xy}=\frac{\delta}{\delta x}\left(\frac{\delta g}{\delta y}\right)=\frac{1}{(x-y)^2}\ now\ we\ find\ \ critical\ point\ , for\ \ this\ g_x=0\ \&g_y=0$$

$$\frac{1}{x-y}+\frac{1-2x}{x(1-x)}=0\ or\frac{1}{x-y}=\frac{2x-1}{x(1-x)}\ \&\ \frac{1-2y}{y(1-y)}-\frac{1}{x-y}=0\ or$$
$$\frac{1}{x-y}=\frac{1-2y}{y(1-y)}$$

$$so\ we\ have\ \frac{2x-1}{x(1-x)}=\frac{1-2y}{y(1-y)}\ or, x(1-x)(1-2y)=y(1-y)(2x-1)$$
$$or, x-x^2-2xy+2x^2y$$
$$=-y+y^2+2x(y-y^2) or, (x+y-1)\big(x(y-1)+y(x-1)\big)=0$$
$$so\ x+y-1=0\ (since\ x,y<1) or, x+y=1\ or, x=1-y$$

$$from\ \frac{1}{x-y}=\frac{2x-1}{x(1-x)}\ we\ get\frac{1}{1-y-y}=\frac{2(1-y)-1}{(1-y)y}\ or, y-y^2=1-4y+4y^2$$

$$or, 5y^2-5y+1=0\ or, y=\frac{5\pm\sqrt{5}}{10}\ as\ x+y=1\ \&\ 1>x>y\ so\ x>\frac{1}{2}$$

$$so\ we\ can\ take\ y=y=\frac{5-\sqrt{5}}{10}=b, x=1-y=y=\frac{5+\sqrt{5}}{10}=a$$

$$now\ g_{xx}at\ (a,b)=\left(-\frac{1}{x^2}-\frac{1}{(1-x)^2}-\frac{1}{(x-y)^2}\right)_{(a,b)}\overset{x=1+y}{=}$$

$$=\left(-\left(\frac{1}{x^2}+\frac{1}{y^2}\right)-\frac{1}{(1-2y)^2}\right)_{a,b}=$$

$$=\left(-\frac{x^2+y^2}{x^2y^2}-\frac{1}{(1-2y)^2}\right)_{a,b}=-15-5=-20(putting\ value\ of\ a\ \&\ b)$$

$$similarly, g_{yy}at\ (a,b)=-20\ \&\ g_{xy}\ at\ (a,b)=5$$

$$now\ maxma\ \ minima\ \ for\ two\ variable$$

$$D=g_{xx}g_{yy}-\left(g_{xy}\right)^2=(-20)(-20)-(5)^2=375>0\ \ and\ g_{xx}=-20<0$$

$$implies\ g(x,y)\ or\ f(x,y)maximum\ at\ (a,b)where\ x=1-y$$

$$P=(1-x)(1-y)(x-y)xy\ or, P_{\max.}\overset{x=1-y}{=}\ y.x.(x-y).xy=(xy)^2(x-y)$$

$$=(ab)^2(a-b)=\left(\frac{5+\sqrt{5}}{10}.\frac{5-\sqrt{5}}{10}\right)^2\left(\frac{5+\sqrt{5}}{10}-\frac{5-\sqrt{5}}{10}\right)=\frac{1}{25}.\frac{1}{\sqrt{5}}=\frac{\sqrt{5}}{125}$$

$$at\ a=x=\frac{5+\sqrt{5}}{10}, b=y=\frac{5-\sqrt{5}}{10}$$

2266.

Let a, b, c be positive real numbers. Prove the following inequalities:

$$\sqrt{a^2+3bc}+\sqrt{b^2+3ca}+\sqrt{c^2+3ab}\geq a+b+c+\sqrt{3(ab+bc+ca)}$$

$$\sqrt{(a^2+3bc)(b^2+3ca)(c^2+3ab)}\geq\left(a+\sqrt{bc}\right)\left(b+\sqrt{ca}\right)\left(a+\sqrt{ab}\right)$$

Proposed by Le Anh Tu-Vietnam

Solutions by Jenish Rijal-Nepal

Solution to problem 1:

$$Let\ p=\sqrt{a^2+3bc}+a,\ q=\sqrt{b^2+3ca}+b, \text{and}\ r=\sqrt{c^2+3ab}+c.$$

$$We\ have: \sqrt{a^2+3bc}-a=\frac{(\sqrt{a^2+3bc}-a)(\sqrt{a^2+3bc}+a)}{\sqrt{a^2+3bc}+a}=\frac{3bc}{p}\ (and\ analogs)$$

$$Thus, it\ suffices\ to\ prove\ that: \sum_{cyc}\frac{3bc}{p}\geq\sqrt{3(ab+bc+ca)}\Leftrightarrow\sum_{cyc}\frac{bc}{p}\geq\sqrt{\frac{ab+bc+ca}{3}}$$

$$Now,\quad \sum_{cyc}\frac{bc}{p}\overset{BERGSTROM}{\geq}\frac{(ab+bc+ca)^2}{\sum bc\cdot p}.$$

$$We\ have:\ \left(\sum_{cyc}bc\cdot p\right)^2\overset{Cauchy}{\leq}\left(\sum_{cyc}bc\right)\left(\sum_{cyc}bc\cdot p^2\right)$$

$$Also,\quad p^2=\left(\sqrt{a^2+3bc}+a\right)^2=2a^2+3bc+2a\sqrt{a^2+3bc}$$

$$So,\quad \sum_{cyc}bc\cdot p^2=2abc(a+b+c)+3\sum_{cyc}b^2c^2+2abc\sum_{cyc}\sqrt{a^2+3bc}$$

$$\sum_{cyc}\sqrt{a^2+3bc}\overset{Cauchy}{\leq}\sqrt{3\sum_{cyc}(a^2+3bc)}=$$

$$=\sqrt{3\left(\sum_{cyc}a\right)^2+3\sum_{cyc}ab}\overset{3\sum ab\leq(\sum a)^2}{\leq}\sqrt{3\left(\sum_{cyc}a\right)^2+\left(\sum_{cyc}a\right)^2}=2\sum_{cyc}a$$

$$\therefore\sum_{cyc}bc\cdot p^2\leq 2abc(a+b+c)+3\sum_{cyc}b^2c^2+2abc\cdot 2\sum_{cyc}a=3(ab+bc+ca)^2$$

$$\therefore\left(\sum_{cyc}bc\cdot p\right)^2\overset{Cauchy}{\leq}(ab+bc+ca)\cdot 3(ab+bc+ca)^2=3(ab+bc+ca)^3$$

$$\sum_{cyc}bc\cdot p\leq\sqrt{3(ab+bc+ca)^3}$$

$$\sum_{cyc}\frac{bc}{p}\overset{BERGSTROM}{\geq}\frac{(ab+bc+ca)^2}{\sum bc\cdot p}\geq\frac{(ab+bc+ca)^2}{\sqrt{3(ab+bc+ca)^3}}=\sqrt{\frac{ab+bc+ca}{3}}$$

Solution to problem 2:

Dividing by abc, we get,

$$\sqrt{\left(\frac{a^2}{bc}+3\right)\left(\frac{b^2}{ca}+3\right)\left(\frac{c^2}{ab}+3\right)}\geq\left(\frac{a}{\sqrt{bc}}+1\right)\left(\frac{b}{\sqrt{ca}}+1\right)\left(\frac{c}{\sqrt{ab}}+1\right)$$

Let $x=\frac{a}{\sqrt{bc}}, y=\frac{b}{\sqrt{ca}}$ *and* $z=\frac{c}{\sqrt{ab}}$. *Notice:* $xyz=1$

It suffices to prove that: $(x^2+3)(y^2+3)(z^2+3) \geq [(x+1)(y+1)(z+1)]^2$

Let $p = x+y+z$ *and* $q = xy+yz+zx$. *Since* $xyz = 1$, *we have,*

$$(x^2+3)(y^2+3)(z^2+3) = x^2y^2z^2 + 3(x^2y^2+y^2z^2+z^2x^2) + 9(x^2+y^2+z^2) + 27$$

$$= 1 + 3(q^2-2p) + 9(p^2-2q) + 27 = 9p^2 + 3q^2 - 6p - 18q + 28$$

$$[(x+1)(y+1)(z+1)]^2 = [xyz + (xy+yz+zx) + (x+y+z) + 1]^2$$

$$= [p+q+2]^2 = p^2 + q^2 + 2pq + 4p + 4q + 4$$

$$(9p^2 + 3q^2 - 6p - 18q + 28) - (p^2 + q^2 + 2pq + 4p + 4q + 4) =$$

$$= 2(4p^2 + q^2 - pq - 5p - 11q + 12) = 2 \cdot \left(\frac{p-2q+3}{2}\right)^2 + \frac{13}{6}(p-3)^2 + \frac{16}{3}(p^2-3q)$$

,which is ≥ 0 *as* $p^2 = (x+y+z)^2 \geq 3(xy+yz+zx) = 3q$.

Equality holds if and only if $a = b = c$ *for both problems.*

2267. If $a, b, c > 0, abc = 1$ and $\lambda \geq 0$ then :

$$\sum_{cyc} \frac{1}{a + \lambda ab} \geq \frac{3}{\lambda + 1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Since $abc = 1$, so we can assign : $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$ and then :

$$\sum_{cyc} \frac{1}{a + \lambda ab} = \sum_{cyc} \frac{1}{\frac{x}{y} + \lambda.\frac{x}{z}} = \sum_{cyc} \frac{(yz)^2}{zx.yz + \lambda.xy.yz} \geq \frac{\left(\sum_{cyc} xy\right)^2}{xyz\left(\sum_{cyc} x\right)(\lambda+1)} \geq$$

$$\frac{3xyz\left(\sum_{cyc} x\right)}{xyz\left(\sum_{cyc} x\right)(\lambda+1)} = \frac{3}{\lambda+1} \quad \forall a, b, c > 0 \,|\, abc = 1, '' = '' \text{ iff } a = b = c = 1 \text{ (QED)}$$

2268. If $x, y, z > 0, \sum_{cyc} x = 1$ and $-3 \leq \lambda \leq \frac{15}{4}$ then :

$$\sum_{cyc} x^3 + \lambda xyz \geq \frac{\lambda + 3}{27}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{An observation : since } \lambda \geq -3 \therefore \sum_{cyc} x^3 + \lambda xyz \geq \sum_{cyc} x^3 - 3xyz$$

$$\overset{AM-GM}{\geq} 0 \text{ and also}, \frac{\lambda+3}{27} \geq 0; \text{now } \sum_{cyc} x^3 + \lambda xyz \overset{?}{\geq} \frac{\lambda+3}{27}$$

$$\Leftrightarrow \sum_{cyc} x^3 - \frac{1}{9}\left(\sum_{cyc} x\right)^3 \overset{?}{\geq} \frac{\lambda}{27}\left(\left(\sum_{cyc} x\right)^3 - 27xyz\right) \text{and} \because$$

$$\left(\sum_{cyc} x\right)^3 - 27xyz \overset{AM-GM}{\geq} 0 \text{ and } \frac{\lambda}{27} \leq \frac{5}{36} \therefore \text{it suffices to prove :}$$

$$4\left(9\sum_{cyc} x^3 - \left(\sum_{cyc} x\right)^3\right) \overset{?}{\geq} 5\left(\left(\sum_{cyc} x\right)^3 - 27xyz\right)$$

$$\Leftrightarrow \sum_{cyc} x^3 + 3xyz \overset{?}{\geq} \sum_{cyc} x^2y + \sum_{cyc} xy^2 \rightarrow \text{true via Schur}$$

$$\therefore \sum_{cyc} x^3 + \lambda xyz \geq \frac{\lambda+3}{27} \; \forall x,y,z > 0 \,\Big|\, \sum_{cyc} x = 1 \text{ and } -3 \leq \lambda \leq \frac{15}{4},$$

$$'' = '' \text{ iff } x = y = z = \frac{1}{3} \text{ (QED)}$$

2269. If $x,y,z > 0, \sum_{cyc} x = 3$ and $3n \geq 4\lambda \geq 0, k = 3n - \lambda$ then :

$$\lambda xyz \geq n\sum_{cyc} xy - k$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\lambda xyz \overset{?}{\geq} n\sum_{cyc} xy - k \Leftrightarrow \lambda xyz \overset{?}{\geq} n\sum_{cyc} xy - 3n + \lambda$$

$$\Leftrightarrow \frac{n}{9}\left(\sum_{cyc} x\right)\left(\left(\sum_{cyc} x\right)^2 - 3\sum_{cyc} xy\right) \overset{?}{\geq} \frac{\lambda}{27}\left(\left(\sum_{cyc} x\right)^3 - 27xyz\right)$$

$$\left(\because \sum_{cyc} x = 3\right) \text{and} \because \frac{n}{9} \geq \frac{4\lambda}{27} \geq 0 \therefore \text{it suffices to prove :}$$

$$4\left(\sum_{cyc} x\right)\left(\left(\sum_{cyc} x\right)^2 - 3\sum_{cyc} xy\right) \overset{?}{\geq} \left(\sum_{cyc} x\right)^3 - 27xyz$$

$$\Leftrightarrow \sum_{cyc} x^3 + 3xyz \overset{?}{\geq} \sum_{cyc} x^2y + \sum_{cyc} xy^2 \rightarrow \text{true via Schur} \therefore \lambda xyz \geq n\sum_{cyc} xy - k$$

$$\forall x, y, z > 0 \mid \sum_{cyc} x = 3 \text{ and } 3n \geq 4\lambda \geq 0, k = 3n - \lambda,$$

$$'' = '' \text{ iff } x = y = z = 1 \text{ (QED)}$$

2270. If $a, b, c > 0, \sum_{cyc} ab = 3$ and $n \in \mathbb{N}$ then :

$$\sum_{cyc}\left(\frac{a^2+b^2}{a^2+b^2+2}\right)^n \geq \frac{3}{2^n}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{a^2+b^2}{a^2+b^2+2} \overset{?}{\geq} \frac{3}{2}$$

$$\Leftrightarrow 3\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) - 3a^2b^2c^2 + 2\left(\sum_{cyc} a^2\right)^2 + 2\sum_{cyc} a^2b^2 \overset{?}{\geq} 8\sum_{cyc} a^2 + 24$$

$$\Leftrightarrow 3\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) - 3a^2b^2c^2 + \frac{2}{3}\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right)^2 +$$

$$\frac{2}{3}\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2b^2\right) \overset{?}{\underset{(*)}{\geq}} \frac{8}{9}\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right)^2 + \frac{24}{27}\left(\sum_{cyc} ab\right)^3 \left(\because \sum_{cyc} ab = 3\right)$$

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s, R, r$ (say); then : $abc = r^2s, \sum_{cyc} a = s, abc = r^2s, \sum_{cyc} ab = 4Rr + r^2, \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2,$

$$\sum_{cyc} a^2b^2 = r^2((4R+r)^2-2s^2), \text{and then} : (*) \Leftrightarrow$$

$$27r^2(s^2-8Rr-2r^2)((4R+r)^2-2s^2)-27r^4s^2+6(4Rr+r^2)(s^2-8Rr-2r^2)^2$$
$$+6r^2(4Rr+r^2)\left((4R+r)^2-2s^2\right) \overset{?}{\geq} 8(s^2-8Rr-2r^2)(4Rr+r^2)^2+8(4Rr+r^2)^3$$

$$\Leftrightarrow (3R-6r)s^4-r(10R^2-43Rr-8r^2)s^2-2r^2(4R+r)^3 \overset{?}{\geq} 0 \text{ and since} : \boxed{\overset{?}{\geq} \text{AUT}\ \textcircled{1}}$$

$$(3R-6r)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen and Euler}}{\geq} 0 \therefore \text{in order to prove } \textcircled{1}, \text{suffices to prove} :$$

$$\text{LHS of } \textcircled{1} \overset{?}{\geq} (3R-6r)(s^2-16Rr+5r^2)^2 \Leftrightarrow$$

$$(86R^2-179Rr+68r^2)s^2 \boxed{\overset{?}{\geq} \text{AUT}\ \textcircled{2}} r(896R^3-1920R^2r+1059Rr^2-148r^3)$$

$$\&\ \text{finally}, (86R^2-179Rr+68r^2)s^2 \overset{\text{Gerretsen}}{\geq} (86R^2-179Rr+68r^2)(16Rr-5r^2)$$
$$\overset{?}{\geq} \text{RHS of } \textcircled{2} \Leftrightarrow 80t^3-229t^2+154t-32 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(80t^2-69t+16) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow \textcircled{2} \Rightarrow \textcircled{1} \text{ is true}$$

$$\therefore \sum_{cyc} \frac{a^2+b^2}{a^2+b^2+2} \geq \frac{3}{2} \therefore \sum_{cyc} \left(\frac{a^2+b^2}{a^2+b^2+2}\right)^n \overset{\text{Holder}}{\geq} \frac{1}{3^{n-1}} \cdot \left(\sum_{cyc} \frac{a^2+b^2}{a^2+b^2+2}\right)^n$$
$$\geq \frac{1}{3^{n-1}} \cdot \left(\frac{3}{2}\right)^n (\because n \in \mathbb{N}) = \frac{3}{2^n} \text{ and so}, \sum_{cyc} \left(\frac{a^2+b^2}{a^2+b^2+2}\right)^n \geq \frac{3}{2^n}$$

$$\forall a,b,c>0 \Big| \sum_{cyc} ab = 3 \text{ and } n \in \mathbb{N}, '' = '' \text{ iff } a=b=c=1 \text{ (QED)}$$

2271. If $x,y,z,t>0, \sum_{cyc} x = 4$ and $\lambda \geq 2 \in \mathbb{N}$ then :

$$\sum_{cyc} \frac{x^2}{x^2+\lambda(y+z+t)} \geq \frac{4}{3\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{x^2}{x^2+\lambda(y+z+t)} = \frac{x^2}{x^2+\lambda(4-x)} \overset{?}{\geq} \frac{7\lambda x-4\lambda+1}{(3\lambda+1)^2}$$
$$\Leftrightarrow x^2(3\lambda+1)^2-\left(x^2+\lambda(4-x)\right)(7\lambda x-4\lambda+1) \overset{?}{\geq} 0$$
$$\Leftrightarrow \lambda\left(16\lambda(x-1)^2-(7x^3-10x^2-x+4)\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow \lambda\left(16\lambda(x-1)^2-(x-1)^2(7x+4)\right)\overset{?}{\geq}0$$

$$\Leftrightarrow \lambda(x-1)^2\left(16(\lambda-2)+7(4-x)\right)\overset{?}{\geq}0$$

$$\Leftrightarrow \lambda(x-1)^2\left(16(\lambda-2)+7(y+z+t)\right)\overset{?}{\geq}0\rightarrow \text{true}\ \because \lambda\geq 2\ \text{and}\ y,z,t>0$$

$$\therefore \frac{x^2}{x^2+\lambda(y+z+t)}\geq\frac{7\lambda x-4\lambda+1}{(3\lambda+1)^2}\ \text{and analogs}\Rightarrow\sum_{cyc}\frac{x^2}{x^2+\lambda(y+z+t)}\geq$$

$$\frac{1}{(3\lambda+1)^2}\cdot\left(7\lambda\left(\sum_{cyc}x\right)-16\lambda+4\right)=\frac{1}{(3\lambda+1)^2}\cdot(7\lambda(4)-16\lambda+4)$$

$$\left(\because\sum_{cyc}x=4\right)=\frac{4(3\lambda+1)}{(3\lambda+1)^2}=\frac{4}{3\lambda+1}\ \text{and so,}\sum_{cyc}\frac{x^2}{x^2+\lambda(y+z+t)}\geq\frac{4}{3\lambda+1}$$

$$\forall x,y,z,t>0\,\Big|\sum_{cyc}x=4\ \text{and}\ \lambda\geq 2\in\mathbb{N},''=''\ \text{iff}\ x=y=z=t=1\ (\text{QED})$$

2272. If $a,b>0;ab=1$ then:

$$\frac{a}{(b^2+1)^3}+\frac{b}{(a^2+1)^3}\geq\frac{1}{4}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{(b^2+1)^3}+\frac{b}{(a^2+1)^3}=\frac{a}{(b^2+ab)^3}+\frac{b}{(a^2+ab)^3}=$$

$$=\frac{a}{b^3(a+b)^3}+\frac{b}{a^3(a+b)^3}=\frac{1}{(a+b)^3}\left(\frac{a}{b^3}+\frac{b}{a^3}\right)=$$

$$=\frac{1}{(a+b)^3}\cdot\frac{a^4+b^4}{(ab)^3}=\frac{a^4+b^4}{(a+b)^3}\cdot\frac{1}{1^3}\overset{BERGSTROM}{\geq}\frac{(a^2+b^2)^2}{2(a+b)^3}\geq\frac{\left(\frac{a^2}{1}+\frac{b^2}{1}\right)^2}{2(a+b)^3}\overset{BERGSTROM}{\geq}$$

$$\geq\frac{\left(\frac{(a+b)^2}{1+1}\right)^2}{2(a+b)^3}=\frac{(a+b)^4}{8(a+b)^3}=\frac{a+b}{8}\overset{AM-GM}{\geq}\frac{2\sqrt{ab}}{8}=\frac{2\cdot\sqrt{1}}{8}=\frac{1}{4}$$

Equality holds for $a=b=1$.

2273. If $a,b,c>0$ then:

$$\sum_{cyc}\frac{(a-b)^2}{a^2+b^2}+3\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^2\leq 3$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$\sum_{cyc}\frac{(a-b)^2}{a^2+b^2}=3-2\sum_{cyc}\frac{ab}{a^2+b^2}=3-2\sum_{cyc}\frac{a^2b^2}{ab(a^2+b^2)}\leq$$

$$\overset{Bergstrom}{\leq}\quad 3-\frac{2(ab+bc+ca)^2}{ab(a^2+b^2)+bc(b^2+c^2)+ca(a^2+c^2)}$$

$$We\ need\ to\ show:$$

$$3-\frac{2(ab+bc+ca)^2}{ab(a^2+b^2)+bc(b^2+c^2)+ca(a^2+c^2)}+3\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^2\leq 3$$

$$3-\frac{2(ab+bc+ca)^2}{(a^2+b^2+c^2)(ab+bc+ca)-abc(a+b+c)}+3\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^2\leq 3$$

$$3(a^2+b^2+c^2)(ab+bc+ca)-3abc(a+b+c)\leq 2(a^2+b^2+c^2)$$

$$2(a^4+b^4+c^4)+4(a^2b^2+b^2c^2+c^2a^2)\geq 3\sum_{cyc}a^3(b+c)+3abc\sum a-3abc\sum a$$

$$\sum(a^4-3a^3b+4a^2b^2-3ab^3+b^4)\geq 0\ or,$$

$$\sum(a-b)^2(a^2-ab+b^2)\geq 0\ True$$

Equality holds for a=b=c.

2274. If $a,b,c>0,\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=2$ then:

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca}+\sqrt{2abc}\leq a+b+c+4abc$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=2\ or, ab+bc+ca+2abc=1$$

$$We\ take\ a=\frac{x}{y+z},b=\frac{y}{z+x},c=\frac{z}{x+y}$$

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca}=\sum\sqrt{ab}=\sum\sqrt{\frac{x}{y+z}.\frac{y}{z+x}}=\sum\sqrt{\frac{y}{y+z}.\frac{x}{z+x}}\leq$$

$$\overset{AM-GM}{\leq}\frac{1}{2}\sum\left(\frac{y}{y+z}+\frac{x}{x+z}\right)=\frac{1}{2}\sum\left(\frac{x}{x+z}+\frac{z}{x+z}\right)=\frac{3}{2}$$

$$(x+y)(y+z)(z+x)\overset{Cesaro}{\geq}8xyz\ or\ \frac{1}{8}\geq\frac{x}{y+z}.\frac{y}{z+x}.\frac{z}{x+y},$$

$$abc\leq\frac{1}{8}\ or\ \sqrt{2abc}\leq\frac{1}{2}$$

$$L.H.S=\sqrt{ab}+\sqrt{bc}+\sqrt{ca}+\sqrt{2abc}\leq\frac{3}{2}+\frac{1}{2}=2$$

$$R.H.S=a+b+c+4abc=\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}+\frac{4xyz}{(x+y)(y+z)(z+x)}$$

$$=\frac{x(x+y)(x+z)+y(y+z)(y+x)+z(z+x)(z+y)+4xyz}{(x+y)(y+z)(z+x)}$$

$$=\frac{\sum x^3+\sum_{sym}x^2y+7xyz}{(x+y)(y+z)(z+x)}=\frac{\sum x^3+\sum_{sym}x^2y+7xyz}{\sum x^2y+2xyz}$$

$$We\ need\ to\ show:$$

$$\frac{\sum x^3+\sum_{sym}x^2y+7xyz}{\sum x^2y+2xyz}\geq 2\ or\sum x^3+3xyz\geq\sum x^2y$$

$$True\ since\ it\ is\ Schur's\ inequality\ of\ degree\ 3$$

$$Equality\ holds\ for\ a=b=c=\frac{1}{2}$$

2275. If $a,b,c>0$, $a^2+b^2+c^2=2(ab+bc+ca)$ then:

$$\frac{\sqrt{bc}}{a+\left(\sqrt{b}-\sqrt{c}\right)^2}+\frac{\sqrt{ca}}{b+\left(\sqrt{c}-\sqrt{a}\right)^2}+\frac{\sqrt{ab}}{c+\left(\sqrt{a}-\sqrt{b}\right)^2}\geq\frac{(a+b+c)^3}{24abc}$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$Let\ x=\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\overset{AM-GM}{\geq}2$$

$$WLOG\ a\ is\ the\ largest\ variable.$$

$$a^2+b^2+c^2=2(ab+bc+ca)\ or,a^2+(b-c)^2-2a(b+c)=0$$

$$or,a^2-2a(b+c)+(b+c)^2=(b+c)^2-(b-c)^2\ or,\left(a-(b+c)\right)^2=4bc$$

$$a-(b+c)=2\sqrt{bc}\ or, a=b+c+2\sqrt{bc}$$

$$a=\left(\sqrt{b}+\sqrt{c}\right)^2\ or, \sqrt{a}=\sqrt{b}+\sqrt{c}\ (A)$$

$$\frac{\sqrt{bc}}{a+\left(\sqrt{b}-\sqrt{c}\right)^2}\overset{by\ (A)}{=}\frac{\sqrt{bc}}{\left(\sqrt{b}+\sqrt{c}\right)^2+\left(\sqrt{b}-\sqrt{c}\right)^2}=\frac{\sqrt{bc}}{2(b+c)}=\frac{1}{2}\frac{1}{\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}}=\frac{1}{2x}$$

$$\frac{\sqrt{ca}}{b+\left(\sqrt{c}-\sqrt{a}\right)^2}=\frac{\sqrt{c}\sqrt{a}}{b+\left(\sqrt{c}-\sqrt{a}\right)^2}\overset{by\ (A)}{=}\frac{\sqrt{c}\left(\sqrt{b}+\sqrt{c}\right)}{b+b}=\frac{1}{2}\left(\frac{c}{b}+\sqrt{\frac{c}{b}}\right)$$

$$\frac{\sqrt{ab}}{c+\left(\sqrt{a}-\sqrt{b}\right)^2}=\frac{\sqrt{a}\sqrt{b}}{c+\left(\sqrt{a}-\sqrt{b}\right)^2}\overset{by\ (A)}{=}\frac{\sqrt{b}\left(\sqrt{b}+\sqrt{c}\right)}{c+c}=\frac{1}{2}\left(\frac{b}{c}+\sqrt{\frac{b}{c}}\right)$$

$$L.H.S=\frac{\sqrt{bc}}{a+\left(\sqrt{b}-\sqrt{c}\right)^2}+\frac{\sqrt{ca}}{b+\left(\sqrt{c}-\sqrt{a}\right)^2}+\frac{\sqrt{ab}}{c+\left(\sqrt{a}-\sqrt{b}\right)^2}$$

$$=\frac{1}{2x}+\frac{1}{2}\left(\frac{c}{b}+\sqrt{\frac{c}{b}}\right)+\frac{1}{2}\left(\frac{b}{c}+\sqrt{\frac{b}{c}}\right)=\frac{1}{2x}+\frac{1}{2}\left(\frac{c}{b}+\frac{b}{c}\right)+\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)=$$

$$=\frac{1}{2x}+\frac{1}{2}(x^2-2)+\frac{x}{2}=\frac{x^3+x^2-2x+1}{2x}$$

$$R.H.S=\frac{(a+b+c)^3}{24abc}=\frac{\left(\left(\sqrt{b}+\sqrt{c}\right)^2+(b+c)\right)^3}{24\left(\sqrt{b}+\sqrt{c}\right)^2bc}=\frac{\left(2(b+c)+2\sqrt{bc}\right)^3}{24bc(b+c+2bc)}=$$

$$=\frac{8}{24}\frac{\left(b+c+\sqrt{bc}\right)^3}{bc(b+c+2bc)}=\frac{1}{3}\frac{\left(\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)+1\right)^3}{\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)+2}=\frac{1}{3}\frac{(x+1)^3}{x+2}$$

We need to show:

$$\frac{x^3+x^2-2x+1}{2x}\geq\frac{1}{3}\frac{(x+1)^3}{x+2}\ or, 3(x+2)(x^3+x^2-2x+1)\geq 2x(x+1)^3$$

$$or, x^4+3x^3-6x^2-11x+6\geq 0\ or, (x-2)(x^3+5x^2+4x-3)\geq 0\ True\ as\ x\geq 2$$

Equalty holds permutation of $(a,b,c)=(4t,t,t)$ *for* $t>0$.

2276. *If* $a,b,c>0$ *then*:

$$a^2+b^2+c^2\geq ab+bc+ca+3\sqrt[3]{\frac{(a-b)^2(b-c)^2(c-a)^2}{4}}$$

Determine all cases of equality

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$WLOG\ a \geq b \geq c\ let\ x = a - b \geq 0, y = b - c \geq 0$$
$$(c - a) = -(a - c) = -\big((a - b) + (b - c)\big) = -(x + y)$$
$$a^2 + b^2 + c^2 - (ab + bc + ca) = \frac{1}{2}((a - b)^2 + (b - c)^2 + (c - a)^2)$$
$$= \frac{1}{2}(x^2 + y^2 + (x + y)^2) = x^2 + xy + y^2$$
$$= \frac{1}{2}(x^2 + y^2) + \frac{1}{2}(x^2 + y^2) + xy \geq$$
$$\overset{AM-GM}{\geq} \frac{1}{2}(x^2 + y^2) + \frac{1}{2} \times 2xy + xy = \frac{1}{2}(x^2 + y^2) + 2xy =$$
$$= \left(\frac{1}{2}x^2 + \frac{1}{2}xy\right) + \left(\frac{1}{2}y^2 + \frac{1}{2}xy\right) + xy = \frac{1}{2}x(x + y) + \frac{1}{2}y(x + y) + xy \overset{AM-GM}{\geq}$$
$$\overset{AM-GM}{\geq} 3\sqrt[3]{\frac{1}{2}x(x + y).\frac{1}{2}y(x + y)xy} = 3\sqrt[3]{\frac{x^2y^2(x + y)^2}{4}}.= 3\sqrt[3]{\frac{(a - b)^2(b - c)^2(c - a)^2}{4}}$$
$$Equality\ x = y\ or\ a - b = b - c\ or\ 2b = a + c, \qquad x = y = 0\ or\ a = b = c.$$

2277. If $x, y, z > 0, xy + yz + zx + xyz = 4$ then:

$$\frac{1 - \sqrt{1 - xyz}}{1 + \sqrt{1 - xyz}} \leq \min\{x, y, z\} \leq 1 - \sqrt{1 - xyz}$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$Let\ x = \frac{2a}{b + c}, y = \frac{2b}{c + a}, z = \frac{2c}{a + b}\ where\ a, b, c > 0$$

$$WLOG\ a \leq b \leq c\ then\ b + c \geq c + a \geq a + b\ then$$
$$x \leq y \leq z\ \&\ \min\{x, y, z\} = x = \frac{2a}{b + c}$$
$$as\ a \leq b \leq c\ and\ b + c \geq c + a \geq a + b\ then\ b + c \geq 2a\ \&\ 0 < x \leq 1$$

$$Let\ P = xyz = \frac{8abc}{(a + b)(b + c)(c + a)} \overset{Cesaro}{\leq} \frac{8abc}{8abc} = 1(A)$$

$$We\ need\ to\ show:$$
$$\min\{x, y, z\} \leq 1 - \sqrt{1 - xyz}\ or, x \leq 1 - \sqrt{1 - xyz}\ or, 1 - xyz \leq (1 - x)^2$$
$$or, 1 - P \leq 1 - 2x + x^2\ or, P \geq x(2 - x) or, xyz \geq x(2 - x) or, yz \geq 2 - x$$
$$or, \left(\frac{2b}{c + a}\right)\left(\frac{2c}{a + b}\right) \geq 2 - \left(\frac{2a}{b + c}\right)\ or, \frac{2bc}{(a + b)(a + c)} \geq \frac{b + c - a}{b + c}$$

$$or, 2bc(b+c) \geq (a+b)(a+c)(b+c-a)$$
$$2bc(b+c) \geq a(b+c)^2 + bc(b+c) - a^3 - abc$$
$$a^3 - a(b+c)^2 + abc + bc(b+c) \geq 0$$
$$a(a+b+c)(a-b-c) + bc(a+b+c) \geq 0$$
$$(a+b+c)(a-b)(a-c) \geq 0$$
$$True\ as\ a \leq b \leq c.$$
$$Equality\ case: holds\ iff\ (a-b)(a-c) = 0\ , since\ a \leq b \leq c$$
$$then\ a = b\ which\ translate\ to\ x = y \leq z$$

$$We\ need\ to\ show:$$
$$\frac{1-\sqrt{1-xyz}}{1+\sqrt{1-xyz}} \leq \min\{x,y,z\}\ or\ \frac{1-\sqrt{1-P}}{1+\sqrt{1-P}} \leq x$$

$$1-x \leq \sqrt{1-P}(1+x) or\ (1-x)^2 \leq (1-P)(1+x)^2$$

$$P(1+x)^2 \leq (1+x)^2 - (1-x)^2\ or\ P(1+x)^2 \leq 4x\ or\ P \leq \frac{4x}{(1+x)^2}$$
$$xyz \leq \frac{4x}{(1+x)^2}$$
$$yz(1+x)^2 \leq 4\ or\ \frac{4bc}{(c+a)(a+b)} \cdot \frac{(b+c+2a)^2}{(b+c)^2} \leq 4$$

$$\frac{bc}{(c+a)(a+b)} \frac{(b+c+2a)^2}{(b+c)^2} \leq 1\ true$$

$$since\ bc(b+c+2a)^2 = bc((b+c)^2 + 4a(b+c) + 4a^2) =$$
$$= bc(b+c)^2 + 4abc(b+c) + 4a^2bc \overset{AM-GM}{\leq}$$
$$\leq bc(b+c)^2 + 4a\frac{(b+c)^2}{4}(b+c) + 4a^2\frac{(b+c)^2}{4} = (b+c)^2(a^2+ab+ac+bc)$$

$$= (b+c)^2(a+b)(a+c)\ then\ \frac{bc}{(c+a)(a+b)} \frac{(b+c+2a)^2}{(b+c)^2} \leq 1)$$

$$Equality\ case: hold\ for\ b = c\ which\ implies\ that\ x \leq y = z$$

2278. If $a, b, c > 0$ then:

$$\frac{(a+b)^2+1}{(a+b)^3+1} + \frac{(a+b)^2+1}{(a+b)^3+1} + \frac{(a+b)^2+1}{(a+b)^3+1} \leq \frac{a+b+c}{\sqrt{2abc}}$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$\boldsymbol{Lemma}$: $\forall\, x > 0$,

$$\frac{x^2+1}{x^3+1} \le \frac{1}{\sqrt{x}}$$

$\boldsymbol{Proof}$:

$$(x^3+1)^2 - x(x^2+1)^2 = x^6 - x^5 - x + 1 = (x-1)(x^5-1) \ge 0$$

$$so\ \frac{(x^3+1)^2}{(x^2+1)^2} \ge x\ or\ \frac{x^2+1}{x^3+1} \le \frac{1}{\sqrt{x}}$$

$$\frac{(a+b)^2+1}{(a+b)^3+1} + \frac{(a+b)^2+1}{(a+b)^3+1} + \frac{(a+b)^2+1}{(a+b)^3+1} = \sum \frac{(a+b)^2+1}{(a+b)^3+1} \le$$

$$\overset{Lemma}{\le} \sum \frac{1}{\sqrt{a+b}} \overset{CBS}{\le} \sqrt{3\left(\sum \frac{1}{a+b}\right)} \overset{AM-HM}{\le} \sqrt{\frac{3}{4}\sum\left(\frac{1}{a}+\frac{1}{b}\right)} = \sqrt{\frac{3}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} =$$

$$= \sqrt{\frac{3}{2}\frac{ab+bc+ca}{abc}} \le \sqrt{\frac{\frac{(a+b+c)^2}{3}}{abc}\cdot\frac{3}{2}} = \frac{a+b+c}{\sqrt{2abc}}$$

$$Equality\ holds\ for\ a=b=c=\frac{1}{2}.$$

2279. If $a, b, c > 0, abc = 1$ then :

$$\sum_{cyc} \frac{1}{\sqrt[4]{(2+a^2)(1+a^2+b^2)}} \le \sqrt{3}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Since } abc = 1, \text{we can assign : } a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x} \text{ and so,}$$

$$\sum_{cyc} \frac{1}{\sqrt[4]{(2+a^2)(1+a^2+b^2)}} = \sum_{cyc} \frac{1}{\sqrt[4]{\left(2+\frac{x^2}{y^2}\right)\left(1+\frac{x^2}{y^2}+\frac{y^2}{z^2}\right)}}$$

$$= \sqrt{3}.\sum_{cyc} \frac{y.\sqrt{z}}{\sqrt[4]{(y^2+y^2+x^2)(x^2z^2+y^2z^2+y^4)(1+1+1)(1+1+1)}}$$

$$\overset{\text{Holder}}{\leq} \sqrt{3}.\sum_{cyc}\frac{y.\sqrt{z}}{\sqrt{xyz}+y.\sqrt{z}+y.\sqrt{x}}=\sqrt{3}.\sum_{cyc}\frac{y.\sqrt{z}}{\sqrt{y}.\left(\sqrt{zx}+\sqrt{yz}+\sqrt{xy}\right)}$$

$$=\sqrt{3}.\sum_{cyc}\frac{\sqrt{yz}}{\sqrt{zx}+\sqrt{yz}+\sqrt{xy}}=\sqrt{3}.\frac{\sqrt{yz}+\sqrt{zx}+\sqrt{xy}}{\sqrt{zx}+\sqrt{yz}+\sqrt{xy}}=\sqrt{3}\ \text{and so,}$$

$$\sum_{cyc}\frac{1}{\sqrt[4]{(2+a^2)(1+a^2+b^2)}}\leq\sqrt{3}\ \forall\ a,b,c>0\ \Big|\ abc=1,$$

$$''=''\ \text{iff}\ a=b=c=1\ (\text{QED})$$

2280.

$$\text{If } a,b,c>0,\sum_{cyc}a^2=3 \text{ and } n\in\mathbb{N} \text{ then :}$$

$$\sum_{cyc}\left(\frac{a^2+b^2}{bc(bc+a^2)}\right)^n\geq 3$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^2+b^2}{bc(bc+a^2)}\overset{AM-GM}{\geq}\sum_{cyc}\frac{2ab}{bc(bc+a^2)}=2.\sum_{cyc}\frac{\left(\sqrt{\frac{a}{c}}\right)^2}{bc+a^2}\overset{\text{Bergstrom}}{\geq}$$

$$2.\frac{\left(\sum_{cyc}\sqrt{\frac{b}{a}}\right)^2}{\sum_{cyc}ab+\sum_{cyc}a^2}\overset{AM-GM}{\geq}2.\frac{\left(3.\sqrt[3]{\sqrt{\frac{b}{a}}.\frac{a}{c}.\frac{c}{b}}\right)^2}{\sum_{cyc}ab+\sum_{cyc}a^2}\geq\frac{18}{2\sum_{cyc}a^2}=3\left(\because\sum_{cyc}a^2=3\right)$$

$$\text{and so, via Holder,}\sum_{cyc}\left(\frac{a^2+b^2}{bc(bc+a^2)}\right)^n\geq\frac{\left(\sum_{cyc}\frac{a^2+b^2}{bc(bc+a^2)}\right)^n}{3^{n-1}}\geq\frac{3^n}{3^{n-1}}$$

$$(\because n\geq 0\ as\ n\in\mathbb{N})=3\ \text{and so,}\sum_{cyc}\left(\frac{a^2+b^2}{bc(bc+a^2)}\right)^n\geq 3$$

$$\forall\ a,b,c>0\ \Big|\ \sum_{cyc}a^2=3\ \text{and}\ n\in\mathbb{N},''=''\ \text{iff}\ a=b=c=1\ (\text{QED})$$

2281. If $x,y,z>0$ then :

$$\left(\frac{\sum_{cyc}xy}{\sum_{cyc}x^2}\right)^2+\frac{\sum_{cyc}x^3}{3xyz}\geq 2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{\sum_{cyc}xy}{\sum_{cyc}x^2}\right)^2+\frac{\sum_{cyc}x^3}{3xyz}\overset{?}{\geq}2$$

$$\Leftrightarrow\frac{\sum_{cyc}x^3-3xyz}{3xyz}\overset{?}{\geq}\frac{\left(\sum_{cyc}x^2-\sum_{cyc}xy\right)\left(\sum_{cyc}x^2+\sum_{cyc}xy\right)}{\left(\sum_{cyc}x^2\right)^2}$$

$$\Leftrightarrow\frac{\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2-\sum_{cyc}xy\right)}{3xyz}\overset{?}{\geq}\frac{\left(\sum_{cyc}x^2-\sum_{cyc}xy\right)\left(\sum_{cyc}x^2+\sum_{cyc}xy\right)}{\left(\sum_{cyc}x^2\right)^2}$$

$$\Leftrightarrow\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2\right)^2\overset{?}{\geq}3xyz\left(\sum_{cyc}x^2+\sum_{cyc}xy\right)\left(\because\sum_{cyc}x^2-\sum_{cyc}xy\geq 0\right)$$

$$\text{and since}:\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2\right)\overset{AM-GM}{\geq}\left(3.\sqrt[3]{xyz}\right)\left(3.\sqrt[3]{x^2y^2z^2}\right)=9xyz$$

$$\therefore\text{it suffices to prove}:9xyz\left(\sum_{cyc}x^2\right)\overset{?}{\geq}3xyz\left(\sum_{cyc}x^2+\sum_{cyc}xy\right)$$

$$\Leftrightarrow 3\sum_{cyc}x^2\overset{?}{\geq}\sum_{cyc}x^2+\sum_{cyc}xy\Leftrightarrow 2\sum_{cyc}x^2\overset{?}{\geq}\sum_{cyc}xy\rightarrow\text{true (strict inequality)}$$

$$\because\sum_{cyc}x^2\geq\sum_{cyc}xy\ \text{and so,}\left(\frac{\sum_{cyc}xy}{\sum_{cyc}x^2}\right)^2+\frac{\sum_{cyc}x^3}{3xyz}\geq 2\ \forall\ x,y,z>0,$$

$$''='' \text{ iff } x=y=z\ (\text{QED})$$

2282. If $a,b,c>0$ and

$3+a+b+c=ab+bc+ca+3abc$ then prove that :

$$2\left(\sqrt{1+a}+\sqrt{1+b}+\sqrt{1+c}\right)\leq 3\sqrt{(1+a)(1+b)(1+c)}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } 1+a=x, 1+b=y, 1+c=z \text{ and then}: 3+\sum_{cyc} a=\sum_{cyc} ab+3abc \Rightarrow$$

$$\sum_{cyc} x=\sum_{cyc}\big(x(y-1)(z-1)\big)=3xyz+\sum_{cyc} x-2\sum_{cyc} xy \Rightarrow 2\sum_{cyc} xy=3xyz \to (*)$$

$$\text{Let } \sqrt{x}=u, \sqrt{y}=v \text{ and } \sqrt{z}=w \text{ and then}: (*) \Rightarrow 2\sum_{cyc} u^2v^2=3u^2v^2w^2 \to (**)$$

$$\text{Now}, 2\left(\sqrt{1+a}+\sqrt{1+b}+\sqrt{1+c}\right) \overset{?}{\leq} 3.\sqrt{(1+a)(1+b)(1+c)}$$

$$\Leftrightarrow 2\sum_{cyc} u \overset{?}{\leq} 3uvw \Leftrightarrow 2\sum_{cyc} u.\frac{3u^2v^2w^2}{2\sum_{cyc} u^2v^2} \overset{?}{\leq} 3uvw \ (\text{via } (**))$$

$$\Leftrightarrow \sum_{cyc} u^2v^2 \overset{?}{\geq} uvw.\sum_{cyc} u \to \text{true}$$

$$\therefore 2\left(\sqrt{1+a}+\sqrt{1+b}+\sqrt{1+c}\right) \leq 3.\sqrt{(1+a)(1+b)(1+c)}$$

$$\forall a,b,c>0 \,\big|\, 3+a+b+c=ab+bc+ca+3abc, '' = '' \text{ iff } a=b=c=1 \ (\text{QED})$$

2283.

If $a,b,c>0$ and $3+a+b+c=ab+bc+ca+3abc$ then prove that :

$$(1+a)(1+b)(1+c) \geq 8$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } 1+a=x, 1+b=y, 1+c=z \text{ and then}: 3+\sum_{cyc} a=\sum_{cyc} ab+3abc \Rightarrow$$

$$\sum_{cyc} x=\sum_{cyc}\big(x(y-1)(z-1)\big)=3xyz+\sum_{cyc} x-2\sum_{cyc} xy \Rightarrow 2\sum_{cyc} xy=3xyz \to (*)$$

$$(1+a)(1+b)(1+c) \overset{?}{\geq} 8 \Leftrightarrow xyz \overset{?}{\geq} \left(\frac{3xyz}{\sum_{cyc} xy}\right)^3 \Leftrightarrow \left(\sum_{cyc} xy\right)^3 \overset{?}{\geq} 27x^2y^2z^2$$

$$\Leftrightarrow \sum_{cyc} xy \overset{?}{\geq} 3.\sqrt[3]{x^2y^2z^2} \to \text{true via } AM-GM \text{ and so}, (1+a)(1+b)(1+c) \geq 8$$

$$\forall a,b,c>0 \,\big|\, 3+a+b+c=ab+bc+ca+3abc, '' = '' \text{ iff } a=b=c=1 \ (\text{QED})$$

2284. If $a,b,c>0$ and

$$3+a+b+c=ab+bc+ca+3abc \text{ then prove that :}$$

$$\frac{1}{3+2a+b}+\frac{1}{3+2b+c}+\frac{1}{3+2c+a}\leq\frac{1}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } 1+a=x,1+b=y,1+c=z \text{ and then : } 3+\sum_{cyc}a=\sum_{cyc}ab+3abc\Rightarrow$$

$$\sum_{cyc}x=\sum_{cyc}\big(x(y-1)(z-1)\big)=3xyz+\sum_{cyc}x-2\sum_{cyc}xy\Rightarrow 2\sum_{cyc}xy=3xyz\to(*)$$

$$\text{Now, }\frac{1}{3+2a+b}+\frac{1}{3+2b+c}+\frac{1}{3+2c+a}=\sum_{cyc}\frac{1}{1+a+1+a+1+b}$$

$$\overset{\text{Reverse Bergstrom}}{\leq}\frac{1}{9}\cdot\sum_{cyc}\left(\frac{2}{1+a}+\frac{1}{1+b}\right)=\frac{1}{3}\cdot\sum_{cyc}\frac{1}{1+a}=\frac{1}{3}\cdot\sum_{cyc}\frac{1}{x}=\frac{1}{3xyz}\cdot\sum_{cyc}xy$$

$$\overset{via\ (*)}{=}\frac{1}{2} \text{ and so, } \frac{1}{3+2a+b}+\frac{1}{3+2b+c}+\frac{1}{3+2c+a}\leq\frac{1}{2}$$

$$\forall a,b,c>0\,\big|\,3+a+b+c=ab+bc+ca+3abc,''=''\text{ iff } a=b=c=1\ (\text{QED})$$

2285. If $a,b,c>0$ and $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=\frac{3}{2}$ then prove that :

$$ab+bc+ca\geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } 1+a=x,1+b=y,1+c=z \text{ and then : } \frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=\frac{3}{2}$$

$$\Rightarrow\sum_{cyc}\frac{1}{x}=\frac{3}{2}\Rightarrow 2\sum_{cyc}xy=3xyz\to(*)$$

$$\text{Now, } ab+bc+ca\overset{?}{\geq}3\Leftrightarrow\sum_{cyc}\big((x-1)(y-1)\big)\overset{?}{\geq}3\Leftrightarrow\sum_{cyc}xy\overset{?}{\geq}2\sum_{cyc}x$$

$$\Leftrightarrow \sum_{cyc} xy \overset{?}{\geq} 2\left(\sum_{cyc} x\right)\cdot\frac{3xyz}{2\sum_{cyc} xy}\ (via\ (*)) \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\geq} 3xyz\left(\sum_{cyc} x\right)$$

$$\rightarrow true\ and\ so, ab+bc+ca \geq 3\ \forall a,b,c>0\ \Big|\ \frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=\frac{3}{2},$$

$$''='' \text{ iff } a=b=c=1\ (QED)$$

2286.

If $a,b,c>0$ and $3+a+b+c=ab+bc+ca+3abc$ then prove that :

$$\frac{1}{3+a^2}+\frac{1}{3+b^2}+\frac{1}{3+c^2}\leq\frac{3}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$Let\ 1+a=x, 1+b=y, 1+c=z\ and\ then: 3+\sum_{cyc} a=\sum_{cyc} ab+3abc \Rightarrow$$

$$\sum_{cyc} x=\sum_{cyc}\big(x(y-1)(z-1)\big)=3xyz+\sum_{cyc} x-2\sum_{cyc} xy \Rightarrow 2\sum_{cyc} xy=3xyz\rightarrow(*)$$

$$Now, \sum_{cyc}\frac{1}{3+a^2}\overset{AM-GM}{\leq}\sum_{cyc}\frac{1}{2(1+a)}=\frac{1}{2}\cdot\sum_{cyc}\frac{1}{x}=\frac{1}{2xyz}\cdot\sum_{cyc} xy\overset{via\ (*)}{=}\frac{3}{4}\ and\ so,$$

$$\frac{1}{3+a^2}+\frac{1}{3+b^2}+\frac{1}{3+c^2}\leq\frac{3}{4}\ \forall a,b,c>0\ \Big|\ 3+a+b+c=ab+bc+ca+3abc,$$

$$''='' \text{ iff } a=b=c=1\ (QED)$$

2287. **If $a,b,c>0$ and $\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}=\frac{3}{2}$ then prove that :**

$$(3a+1)(3b+1)(3c+1)\geq 64$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$Let\ 1+a=x, 1+b=y, 1+c=z\ and\ then: \frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=\frac{3}{2}$$

$$\Rightarrow\sum_{cyc}\frac{1}{x}=\frac{3}{2}\Rightarrow 2\sum_{cyc} xy=3xyz\rightarrow(*)$$

$$Now, (3a+1)(3b+1)(3c+1)\overset{?}{\geq}64\Leftrightarrow(3x-2)(3y-2)(3z-2)\overset{?}{\geq}64$$

$$\Leftrightarrow 9xyz-6\sum_{cyc}xy+4\sum_{cyc}x-24\overset{?}{\geq}0$$

$$\Leftrightarrow 6\sum_{cyc}xy-6\sum_{cyc}xy+4\sum_{cyc}x-24.\frac{3xyz}{2\sum_{cyc}xy}\overset{?}{\geq}0\ (via\ (*))$$

$$\Leftrightarrow\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)\overset{?}{\geq}9xyz\rightarrow true\ \because$$

$$\sum_{cyc}x\overset{AM-GM}{\geq}3.\sqrt[3]{xyz}\ and\ \sum_{cyc}xy\overset{AM-GM}{\geq}3.\sqrt[3]{x^2y^2z^2}$$

$$\therefore (3a+1)(3b+1)(3c+1)\geq 64\ \forall\ a,b,c>0\ \Big|\ \frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=\frac{3}{2},$$

$$''=''\ iff\ a=b=c=1\ (QED)$$

2288. If $a,b,c>0$ then:

$$\sum_{cyc}\frac{a^2+b^2}{a^2+b^2+2c^2}\geq\frac{3}{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Omar Shah-India

Denote:

$$x=a^2+b^2+c^2$$

$$\sum_{cyc}\frac{a^2+b^2}{a^2+b^2+2c^2}=\sum_{cyc}\frac{x-c^2}{x+c^2}=\sum_{cyc}\frac{-x-c^2+2x}{x+c^2}=$$

$$=-3+2x\sum_{cyc}\frac{1}{x+c^2}\overset{BERGSTROM}{\geq}-3+2x\cdot\frac{(1+1+1)^2}{3x+a^2+b^2+c^2}=$$

$$=-3+2x\cdot\frac{9}{3x+x}=-3+\frac{9}{2}=\frac{3}{2}$$

Equality holds for $a=b=c$.

2289.

If $a,b,c>0$ then prove that:

$$\sqrt{1+\frac{8bc}{a^2}}+\sqrt{1+\frac{8ca}{b^2}}+\sqrt{1+\frac{8ab}{c^2}}\geq 3\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$(L.H.S)^2=\sum_{cyc}\left(1+\frac{8bc}{a^2}\right)+2\sum_{cyc}\sqrt{\left(1+\frac{8bc}{a^2}\right)\left(1+\frac{8ca}{b^2}\right)}\overset{c-s}{\geq}$$

$$\geq\sum_{cyc}\left(1+\frac{8bc}{a^2}\right)+2\sum_{cyc}\left(1+\frac{8c}{\sqrt{ab}}\right)=9+8\sum_{cyc}\frac{bc}{a^2}+16\sum_{cyc}\frac{c}{\sqrt{ab}}$$

$$(R.H.S)^2=9(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=27+9\sum_{sym}\left(\frac{a}{b}+\frac{b}{a}\right)$$

$$we\ need\ to\ show:\ 9+8\sum_{cyc}\frac{bc}{a^2}+16\sum_{cyc}\frac{c}{\sqrt{ab}}\geq\ 27+9\sum_{sym}\left(\frac{a}{b}+\frac{b}{a}\right)$$

$$or,8\sum_{cyc}\frac{x^2}{yz}+16\sum_{cyc}\frac{\sqrt{xy}}{z}\geq 18+9\sum_{sym}\left(\frac{x}{y}+\frac{y}{x}\right)\ \left(let\ x=\frac{1}{a},y=\frac{1}{b},z=\frac{1}{c}\right)$$

$$or,8\sum_{cyc}x^3+16\sum_{cyc}(xy)^{\frac{3}{2}}\geq 18xyz+9\sum_{sym}x^2y\ \ (A)$$

$$now,2(xy)^{\frac{3}{2}}+z^3\overset{AM-GM}{\geq}3\sqrt[3]{(xy)^3,z^3}=3xyz\ then\ \sum_{cyc}x^3+2\sum_{cyc}(xy)^{\frac{3}{2}}\geq 9xyz$$

$$x^3+y^3+4(xy)^{\frac{3}{2}}=\left(x^3+(xy)^{\frac{3}{2}}+(xy)^{\frac{3}{2}}\right)+\left(y^3+(xy)^{\frac{3}{2}}+(xy)^{\frac{3}{2}}\right)\overset{AM-GM}{\geq}3x^2y+3y^2x$$

$$then\ 2\sum_{cyc}x^3+4\sum_{cyc}(xy)^{\frac{3}{2}}\geq 3\sum_{sym}x^2y$$

$$now\ from\ (A)\ we\ get$$

$$:8\sum_{cyc}x^3+16\sum_{cyc}(xy)^{\frac{3}{2}}=2\left(\sum_{cyc}x^3+2\sum_{cyc}(xy)^{\frac{3}{2}}\right)+3\left(2\sum_{cyc}x^3+4\sum_{cyc}(xy)^{\frac{3}{2}}\right)\geq$$

$$\geq 2\times 9xyz+3\times 3\sum_{sym}x^2y=18xyz+9\sum_{sym}x^2y\ proof\ complete.$$

$$Equality\ holds\ for\ a=b=c.$$

2290. *If $x,y,z>0,x+y+z=3$ and $\lambda\geq 0$ then:*

$$\sum_{cyc}\frac{x^4}{\sqrt{x^2+\lambda x}}\geq\frac{3}{\sqrt{\lambda+1}}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{x^4}{\sqrt{x^2+\lambda x}}=\sum_{cyc}\frac{x^{3,5}}{\sqrt{x+\lambda}}\overset{HOLDER}{\geq}\frac{(\sum x)^{3,5}}{3^{1,5}\sum\sqrt{x+\lambda}}\overset{CBS}{\geq}$$

$$\geq \frac{(\sum x)^{3,5}}{3^{1,5}\sqrt{3\sum(x+\lambda)}} = \frac{3^{3,5}}{3^{1,5}\sqrt{3(x+y+z+3\lambda)}} = \frac{9}{\sqrt{3(3+3\lambda)}} = \frac{3}{\sqrt{\lambda+1}}$$

$$\textit{Equality holds for } x = y = z = 1.$$

2291. If $a,b,c>0, a+b+c=1$ and $\lambda \geq 0$ then :

$$\left(\sum_{cyc} ab\right)\left(\sum_{cyc} \frac{a}{b^2+\lambda b}\right) \geq \frac{3}{3\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{a}{\sqrt{b+\lambda}} = \sum_{cyc} \frac{a^2}{a.\sqrt{b+\lambda}} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc}\left(a.\sqrt{b+\lambda}\right)} = \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc}\left(\sqrt{a}.\sqrt{ab+\lambda a}\right)}$$

$$\overset{\text{CBS}}{\geq} \frac{\left(\sum_{cyc} a\right)^2}{\sqrt{\sum_{cyc} a}.\sqrt{\sum_{cyc} ab+\lambda\sum_{cyc} a}} \geq \frac{\left(\sum_{cyc} a\right)^2}{\sqrt{\sum_{cyc} a}.\sqrt{\frac{\left(\sum_{cyc} a\right)^2}{3}+\lambda\sum_{cyc} a}} = \frac{1}{\sqrt{\frac{1}{3}+\lambda}}$$

$$\left(\because \sum_{cyc} a = 1\right) \therefore \sum_{cyc} \frac{a}{\sqrt{b+\lambda}} \geq \sqrt{\frac{3}{3\lambda+1}} \to (1) \text{ and now, } \left(\sum_{cyc} ab\right)\left(\sum_{cyc} \frac{a}{b^2+\lambda b}\right)$$

$$\overset{\text{Reverse CBS}}{\geq} \left(\sum_{cyc} \frac{a}{\sqrt{b+\lambda}}\right)^2 \overset{\text{via } (1)}{\geq} \frac{3}{3\lambda+1} \text{ and so, } \left(\sum_{cyc} ab\right)\left(\sum_{cyc} \frac{a}{b^2+\lambda b}\right) \geq \frac{3}{3\lambda+1}$$

$$\forall a,b,c>0 \,|\, a+b+c=1 \text{ and } \lambda \geq 0, '' = '' \text{ iff } a=b=c=\frac{1}{3} \text{ (QED)}$$

2292. If $a,b,c>0, \sum_{cyc} a^2 = 3$ and $\lambda \geq 0$ then :

$$\sum_{cyc} \frac{a}{a+\lambda bc} \leq \frac{3}{1+\lambda abc}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{a}{a+\lambda bc} = \sum_{cyc} \frac{a+\lambda bc-\lambda bc}{a+\lambda bc} = 3-\lambda abc.\sum_{cyc} \frac{1}{a^2+\lambda abc} \overset{\text{Bergstrom}}{\leq}$$

$$3-\frac{9\lambda abc}{\sum_{cyc}a^2+3\lambda abc}\overset{\sum_{cyc}a^2=3}{=}3-\frac{9\lambda abc}{3+3\lambda abc}=\frac{3}{1+\lambda abc}\ \text{and so,}$$

$$\sum_{cyc}\frac{a}{a+\lambda bc}\le\frac{3}{1+\lambda abc}\ \forall a,b,c>0\ \Big|\ \sum_{cyc}a^2=3\ \text{and}\ \lambda\ge 0,$$

$$''='' \text{ iff } a=b=c=1\ (\text{QED})$$

2293. If $a,b,c>0,\sum_{cyc}a=6$ and $\lambda\ge 1$ then :

$$\sum_{cyc}\frac{a^2}{\lambda bc+\sqrt{a^3+8}}\ge\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^2}{\lambda bc+\sqrt{a^3+8}}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}a\right)^2}{\lambda\sum_{cyc}bc+\sum_{cyc}\sqrt{(a+2)(a^2-2a+4)}}$$

$$\overset{AM-GM}{\ge}\frac{\left(\sum_{cyc}a\right)^2}{\lambda\sum_{cyc}ab+\frac{\sum_{cyc}a^2-\sum_{cyc}a+18}{2}}\overset{\sum_{cyc}a=6}{=}\frac{2\left(\sum_{cyc}a\right)^2}{2\lambda\sum_{cyc}ab+\sum_{cyc}a^2-\frac{\left(\sum_{cyc}a\right)^2}{6}+\frac{\left(\sum_{cyc}a\right)^2}{2}}$$

$$=\frac{12\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)}{12\lambda\sum_{cyc}ab+6\sum_{cyc}a^2+2\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)}\overset{?}{\ge}\frac{3}{\lambda+1}$$

$$\Leftrightarrow 4(\lambda-1)\sum_{cyc}a^2\overset{?}{\ge}4(\lambda-1)\sum_{cyc}ab\Leftrightarrow 4(\lambda-1)\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)\overset{?}{\ge}0\to\text{true}$$

$$\because\lambda\ge 1\ \text{and so,}\ \sum_{cyc}\frac{a^2}{\lambda bc+\sqrt{a^3+8}}\ge\frac{3}{\lambda+1}\ \forall a,b,c>0\ \Big|\ \sum_{cyc}a=6\ \text{and}\ \lambda\ge 1,$$

$$''='' \text{ iff } a=b=c=2\ (\text{QED})$$

2294. *If $a,b,c>0,a+b+c=2$ then:*

$$\sqrt{2}\left(\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}\right)-\sqrt{11+\frac{4}{abc}}\ge\sqrt{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

Let $x^2=a,y^2=b,z^2=c$ then $x^2+y^2+z^2=2$

Let $p=xyz,q=xy+yz+zx,r=x+y+z$

$$x^2+y^2+z^2=2\ or,3(xyz)^{\frac{2}{3}}\overset{AM-GM}{\leq}2\ or,xyz\leq\left(\frac{2}{3}\right)^{\frac{3}{2}}\ or,p\leq\left(\frac{2}{3}\right)^{\frac{3}{2}}(1)$$

$$r^2=(x+y+z)^2=\sum x^2+2\sum xy=2+2q\ (2)\ \&\ r^2=(x+y+z)^2\leq 3\sum x^2=6\ (3)$$

$$q=xy+yz+zx\overset{AM-GM}{\geq}3(xyz)^{\frac{2}{3}}=3r^{\frac{2}{3}}\ then\ 4q\geq 12p^{\frac{2}{3}}\ (4)$$

$$\left(\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}\right)=\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}=\frac{r}{p}\ \&\ \sqrt{11+\frac{4}{abc}}=\sqrt{11+\frac{4}{p^2}}$$

We need to show:

$$\sqrt{2}\left(\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}\right)-\sqrt{11+\frac{4}{abc}}\geq\sqrt{2}$$

$$or,\frac{\sqrt{2}r}{p}-\sqrt{11+\frac{4}{p^2}}\geq\sqrt{2}\ or,\sqrt{2}(r-p)\geq\sqrt{4+11p^2}$$

$$2(r-p)^2\geq 4+11p^2$$

$$or,2r^2-4rp-9p^2\geq 4\ or,2(2+2q)-4rp-9p^2\overset{(2)}{\geq}4$$

$$4q\geq 4rp+9p^2$$

$$12p^{\frac{2}{3}}\overset{(4)\ \&\ (3)}{\geq}4\sqrt{6}p+9p^2$$

$$12\geq 4\sqrt{6}p^{\frac{1}{3}}+9p^{\frac{4}{3}}\ true$$

$$\left(as\ 4\sqrt{6}p^{\frac{1}{3}}+9p^{\frac{4}{3}}\overset{(1)}{\leq}4\sqrt{6}\times\left(\frac{2}{3}\right)^{\frac{1}{2}}+9\left(\frac{2}{3}\right)^2=4\times 2+4=12\right)$$

$$Equality\ holds\ for\ a=b=c=\frac{2}{3}$$

2295. *If $a,b,c>0, a+b+c=3$ then prove that*:

$$3(a^2+b^2+c^2)+\frac{ab+bc+ca}{a^2+b^2+c^2}\geq 10$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$m=a^2+b^2+c^2=\frac{a^2}{1}+\frac{b^2}{1}+\frac{c^2}{1}\overset{Radon}{\geq}\frac{(a+b+c)^2}{3}\overset{a+b+c=3}{=}3$$

$$(a^2+b^2+c^2)+2(ab+bc+ca)=(a+b+c)^2\overset{a+b+c=3}{=}9$$

$$or,(ab+bc+ca)=\frac{9-(a^2+b^2+c^2)}{2}$$

$$now\ L.H.S = 3(a^2+b^2+c^2)+\frac{9-(a^2+b^2+c^2)}{2(a^2+b^2+c^2)} \overset{m=a^2+b^2+c^2}{=} 3m+\frac{9-m}{2m}$$

$$We\ need\ to\ show: 3m+\frac{9-m}{2m} \geq 10$$

$$6m^2-21m+9 \geq 0\ or\ 3(m-3)(2m-1) \geq 0$$

$$True\ as\ m \geq 3\ , Equality\ holds\ for\ a=b=c=1$$

2296. *If* $a,b,c>2, abc=2(a+b+c)$ *then*

$$(a-2)(b-2)(c-2) \leq 18\sqrt{6}-44$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ x=a-2, y=b-2, z=c-2\ , a,b,ç>2\ then\ x,y,z>0$$

$$abc=2(a+b+c)\ or, (x+2)(y+2)(z+2)=2(x+y+z+6)$$

$$or, xyz+2(xy+yz+zx)+4(x+y+z)+8=2(x+y+z)+12$$

$$or, xyz+2\times 3(xyz)^{\frac{2}{3}}+2\times 3(xyz)^{\frac{1}{3}} \overset{AM-GM}{\leq} 4\ or, t^3+6t^2+6t \overset{xyz=t^3}{\leq} 4$$

$$or, (t+2)(t^2+4t-2) \leq 0\ now, t^3=xyz>0\ then\ t+2>0\ which\ indicate\ that$$

$$(t^2+4t-2) \leq 0\ or, (t+2)^2 \leq 6\ or, t+2 \leq \sqrt{6}\ or, t \leq \sqrt{6}-2$$

$$(a-2)(b-2)(c-2)=xyz=t^3 \leq \left(\sqrt{6}-2\right)^3 = 18\sqrt{6}-44$$

$$Equality\ holds\ for\ a=b=c=\sqrt{6}$$

2297. *If* $a,b,c>0$ *and* $a+b+c=3$ *then*:

$$a^4b+b^4c+c^4a+18 \geq 7(ab+bc+ca)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$x^4+1+1+1 \overset{AM-GM}{\geq} 4x\ or, x^4 \geq 4x-3\ (1)$$

$$(a+b+c)^2 \geq 3(ab+b+ca) or, ab+bc+ca \overset{a+b+c=3}{\leq} 3\ (2)$$

$$a^4b+b^4c+c^4a+18 \overset{(1)}{\geq} 4(ab+bc+ca)-9+18=4(ab+bc+ca)+9$$

$$\geq 4(ab+bc+ca)+3(ab+bc+ca)=7(ab+bc+ca)$$

$$Equality\ holds\ for\ a=b=c=1$$

2298. *If* $a,b,c,d>0$ *and* $\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4}=1$

then prove that $abcd \geq 3$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ x=\frac{1}{1+a^4}, y=\frac{1}{1+b^4}, z=\frac{1}{1+c^4}, w=\frac{1}{1+d^4}\ then\ x+y+z+w=1$$

$$1+a^4=\frac{1}{x}\ or, a^4=\frac{1}{x}-1=\frac{1-x}{x}\overset{x+y+z+w=1}{=}\frac{y+z+w}{x}\overset{AM-GM}{\geq}\frac{3\sqrt[3]{yzw}}{x}$$

$$similarly, b^4\geq\frac{3\sqrt[3]{xzw}}{y}, c^4\geq\frac{3\sqrt[3]{yxw}}{z}, d^4\geq\frac{3\sqrt[3]{yzx}}{w}$$

$$Now\ a^4b^4c^4d^4\geq 81 or, abcd\geq 3$$

$$Equality\ holds\ for\ a=b=c=d=\sqrt[4]{3}$$

2299. *If $a,b,c>0$ then prove*

$$\frac{a^2+b^2}{a^2+b^2+2c^2}+\frac{b^2+c^2}{b^2+c^2+2a^2}+\frac{c^2+a^2}{c^2+a^2+2b^2}\geq\frac{3}{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{a^2+b^2}{a^2+b^2+2c^2}+\frac{b^2+c^2}{b^2+c^2+2a^2}+\frac{c^2+a^2}{c^2+a^2+2b^2}\geq\frac{3}{2}$$

$$Let\ a^2+b^2=x>0,\quad b^2+c^2=y>0,\quad c^2+a^2=z>0. Then:$$

$$a^2=\frac{z+x-y}{2},\ b^2=\frac{x+y-z}{2},\ c^2=\frac{z+y-x}{2}$$

$$\frac{x}{x+2c^2}+\frac{y}{y+2a^2}+\frac{z}{z+2b^2}=$$

$$\frac{x}{x+z+y-x}+\frac{y}{y+z+x-y}+\frac{z}{z+x+y-z}=$$

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\overset{Nesbitt}{\geq}\frac{3}{2}$$

$$Equality\ holds\ for\ a=b=c.$$

2300.

$$\text{If } x,y,z>0, \sum_{cyc} x=1 \text{ then :}$$

$$\sum_{cyc}\sqrt{\frac{yz}{1-x^2}}\leq\frac{3}{2\sqrt{2}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z>0, b+c-a=2x>0$ and $c+a-b=2y>0 \Rightarrow a+b>c, b+c>a, c+a>b \Rightarrow a,b,c$ form sides of a triangle ABC with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc} x=\sum_{cyc} a=2s \Rightarrow \sum_{cyc} x=s \Rightarrow \sum_{cyc} xy=4Rr+r^2$ and so,

$$\sum_{cyc}\sqrt{\frac{yz}{1-x^2}} \overset{CBS}{\leq} \sqrt{\sum_{cyc} xy}\cdot\sqrt{\sum_{cyc}\frac{1}{1-x^2}} \overset{\sum_{cyc} x=1}{=} \sqrt{\sum_{cyc} xy}\cdot\sqrt{\sum_{cyc}\frac{1}{(y+z)(x+y+z+x)}}$$

$$=\sqrt{4Rr+r^2}\cdot\sqrt{\sum_{cyc}\frac{1}{a(b+c)}}$$

$$=\sqrt{4Rr+r^2}\cdot\sqrt{\frac{1}{4Rrs.\,2s(s^2+2Rr+r^2)}\cdot\sum_{cyc}\left(bc\left(a^2+\sum_{cyc} ab\right)\right)}$$

$$=\sqrt{4Rr+r^2}\cdot\sqrt{\frac{8Rrs^2+(s^2+4Rr+r^2)^2}{8Rrs^2(s^2+2Rr+r^2)}} \overset{?}{\leq} \frac{3}{2\sqrt{2}}$$

$$\Leftrightarrow (5R-r)s^4-r(46R^2+15Rr+2r^2)s^2-r^2(4R+r)^3 \boxed{\overset{?}{\geq}} 0 \text{ and } \because \boxed{①}$$

$$(5R-r)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } ①, \text{it suffices to prove :}$$

$$\text{LHS of } ① \overset{?}{\geq} (5R-r)(s^2-16Rr+5r^2)^2 \Leftrightarrow$$

$$(114R^2-97Rr+8r^2)s^2 \boxed{\overset{?}{\geq}} r(1344R^3-1008R^2r+297Rr^2-24r^3) \text{ and finally,}$$

$$(114R^2-97Rr+8r^2)s^2 \overset{\text{Gerretsen}}{\geq} (114R^2-97Rr+8r^2)(16Rr-5r^2) \overset{?}{\geq}$$

$$\text{RHS of } ② \Leftrightarrow 240R^3-557R^2r+158Rr^2-8r^3 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R-2r)(114R^2-77Rr+4r^2) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow ② \Rightarrow ① \text{ is true}$$

$$\therefore \sum_{cyc}\sqrt{\frac{yz}{1-x^2}} \leq \frac{3}{2\sqrt{2}} \ \forall x,y,z>0, \sum_{cyc} x=1, '' = '' \text{ iff } x=y=z=\frac{1}{3} \text{ (QED)}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru