ROMANIAN MATHEMATICAL MAGAZINE

JP.645 Solve for real numbers:

$$\begin{cases} x,y,z>0 \\ x+y+z=1 \\ x\sqrt{3y}+y\sqrt{3z}+z\sqrt{3x}=1 \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$x\sqrt{3y}+y\sqrt{3z}+z\sqrt{3x}=\sqrt{3}\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)=$$

$$=\sqrt{3}\left(\sqrt{x}\cdot\sqrt{xy}+\sqrt{y}\cdot\sqrt{yz}+\sqrt{z}\cdot\sqrt{zx}\right)\overset{CBS}{\leq}\sqrt{3}\cdot\sqrt{(x+y+z)(xy+yz+zx)}=$$

$$=\sqrt{3}\cdot\sqrt{xy+yz+zx}\leq\sqrt{3}\cdot\sqrt{\frac{1}{3}(x+y+z)^2}=\sqrt{3}\cdot\frac{1}{\sqrt{3}}\cdot(x+y+z)=1$$

$$x\sqrt{3y}+y\sqrt{3z}+z\sqrt{3x}\leq 1$$

Equality holds for: $x=y=z=\frac{1}{3}$.

Solution 2 by Marin Chirciu-Romania

$$LHS=x\sqrt{3y}+y\sqrt{3z}+z\sqrt{3x}=\sum\sqrt{x}\sqrt{3xy}\overset{SOS}{\leq}\sqrt{\sum x\left(\sum x\right)^2}=$$

$$=\sqrt{1\cdot 1^2}=1=RHS,\ \text{with equal for}\ \frac{\sqrt{x}}{\sqrt{3xy}}=\frac{\sqrt{y}}{\sqrt{3yz}}=\frac{\sqrt{z}}{\sqrt{3zx}}\ \text{and}\ x+y+z=1\Leftrightarrow$$

$$\Leftrightarrow(x,y,z)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right).$$

We deduce that $(x,y,z)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ **is the unique solution of the system.**

Remark.

Solve for real numbers $\begin{cases} x,y,z,t>0 \\ x+y+z+t=1 \\ x\sqrt{\frac{8}{3}y}+y\sqrt{\frac{8}{3}z}+z\sqrt{\frac{8}{3}t}+t\sqrt{\frac{8}{3}x}=1 \end{cases}$

Marin Chirciu

Solution

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$$LHS = x\sqrt{\frac{8}{3}y} + y\sqrt{\frac{8}{3}z} + z\sqrt{\frac{8}{3}t} + t\sqrt{\frac{8}{3}x} = \sum \sqrt{x}\sqrt{\frac{8}{3}xy} \overset{CBS}{\leq} \sqrt{\sum x \sum \frac{8}{3}xy} \overset{SOS}{\leq}$$

$$\leq \sqrt{\sum x (\sum x)^2} = \sqrt{1 \cdot 1^2} = 1 = RHS, \text{with equal for}$$

$$\frac{\sqrt{x}}{\sqrt{\frac{8}{3}xy}} = \frac{\sqrt{y}}{\sqrt{\frac{8}{3}yz}} = \frac{\sqrt{z}}{\sqrt{\frac{8}{3}zt}} = \frac{\sqrt{t}}{\sqrt{\frac{8}{3}tx}} \text{ and } x + y + z + t = 1 \Leftrightarrow (x, y, z, t) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

$$\frac{8}{3}\sum xy \leq \left(\sum x\right)^2 \Leftrightarrow \sum (x - y)^2 \geq 0.$$

We denoted above $\sum xy = xy + xz + xt + yz + yt + zt$ and

$$\sum (x - y)^2 = (x - y)^2 + (x - z)^2 + (x - t)^2 + (y - z)^2 + (y - t)^2 + (z - t)^2$$

We deduce that $(x, y, z, t) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ is the unique solution of the system.

Remark.

Solve for real numbers $\begin{cases} x, y, z, t > 0 \\ x^2 + y^2 + z^2 + t^2 = 1 \\ x\sqrt{\frac{3}{2}y} + y\sqrt{\frac{3}{2}z} + z\sqrt{\frac{3}{2}t} + t\sqrt{\frac{3}{2}x} = 1 \end{cases}$

Marin Chirciu

Solution

$$LHS = \sum \sqrt{x}\sqrt{\frac{3}{2}xy} \overset{CBS}{\leq} \sqrt{\sum x \sum \frac{3}{2}xy} \overset{SOS}{\leq} \sqrt{\sum x \left(\sum x^2\right)} = ???$$

$$= \sqrt{1 \cdot 1^2} = 1 = RHS, \text{ with equal for } \frac{\sqrt{x}}{\sqrt{\frac{8}{3}xy}} = \frac{\sqrt{y}}{\sqrt{\frac{8}{3}yz}} = \frac{\sqrt{z}}{\sqrt{\frac{8}{3}zt}} = \frac{\sqrt{t}}{\sqrt{\frac{8}{3}tx}} \text{ and } x + y + z + t = 1 \Leftrightarrow$$

$$\Leftrightarrow (x, y, z, t) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

$$\frac{8}{3}\sum xy \leq \left(\sum x\right)^2 \Leftrightarrow \sum (x - y)^2 \geq 0.$$

We denoted above $\sum xy = xy + xz + xt + yz + yt + zt$ and

$$\sum (x - y)^2 = (x - y)^2 + (x - z)^2 + (x - t)^2 + (y - z)^2 + (y - t)^2 + (z - t)^2$$

We deduce that $(x, y, z, t) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ is the unique solution of the system.

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\begin{cases} x,y,z>0 \\ x+y+z=1 \\ x\sqrt{3y}+y\sqrt{3z}+z\sqrt{3x}=1 \end{cases}$$

$$x\sqrt{3y}+y\sqrt{3z}+z\sqrt{3x}=1 \Leftrightarrow \sqrt{3}\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)=1$$

$$\Leftrightarrow \left[\sqrt{3}\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)\right]^2=1^2 \Leftrightarrow 3\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)^2=1$$

$$\left(\sqrt{x}\cdot\sqrt{xy}+\sqrt{y}\cdot\sqrt{yz}+\sqrt{z}\cdot\sqrt{zx}\right)^2 \leq$$

$$\leq \left(\left(\sqrt{x}\right)^2+\left(\sqrt{y}\right)^2+\left(\sqrt{z}\right)^2\right)\left(\left(\sqrt{xy}\right)^2+\left(\sqrt{yz}\right)^2+\left(\sqrt{zx}\right)^2\right)$$

$$\Leftrightarrow \left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)^2 \leq (x+y+z)(xy+yz+zx)$$

$$\Rightarrow 3\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)^2 \leq 3(x+y+z)(xy+yz+zx)$$

$$x+y+z=1 \wedge 3\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)^2=1 \Rightarrow 1 \leq 3(1)(xy+yz+zx) \Leftrightarrow$$

$$\Leftrightarrow 3(xy+yz+zx) \geq 1 \quad \textbf{(i)}$$

$$(x-y)^2+(y-z)^2+(z-x)^2 \geq 0$$

$$\Leftrightarrow x^2-2xy+y^2+y^2-2yz+z^2+z^2-2zx+x^2 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow 2(x^2+y^2+z^2)-2(xy+yz+zx) \geq 0$$

$$\Leftrightarrow x^2+y^2+z^2 \geq xy+yz+zx$$

$$\Leftrightarrow (x^2+y^2+z^2)+2(xy+yz+zx) \geq (xy+yz+zx)+2(xy+yz+zx) \Leftrightarrow$$

$$\Leftrightarrow (x+y+z)^2 \geq 3(xy+yz+zx)$$

$$x+y+z=1 \Rightarrow 1^2 \geq 3(xy+yz+zx) \Leftrightarrow 3(xy+yz+zx) \leq 1 \quad \textbf{(ii)}$$

$$\begin{cases} 3(xy+yz+zx) \geq 1 \\ 3(xy+yz+zx) \leq 1 \end{cases} \Rightarrow 3(xy+yz+zx)=1 \Rightarrow (x+y+z)^2=3(xy+yz+zx)$$

$$\Leftrightarrow 1=3(xy+yz+zx) \Leftrightarrow x^2+y^2+z^2+2xy+2yz+2zx=3xy+3yz+3zx$$

$$\Leftrightarrow x^2+y^2+z^2-xy-yz-zx=0 \Leftrightarrow \frac{1}{2}[(x-y)^2+(y-z)^2+(z-x)^2]=0$$

$$\Leftrightarrow (x-y)^2+(y-z)^2+(z-x)^2=0 \Leftrightarrow x-y=0 \wedge y-z=0 \wedge z-x=0 \Leftrightarrow$$

$$\Leftrightarrow x=y=z$$

$$x+y+z=1 \wedge x=y=z \Rightarrow 3x=1 \Leftrightarrow x=\frac{1}{3} \Rightarrow x=y=z=\frac{1}{3}$$

$$\left(\frac{1}{3}\right)\sqrt{3\cdot\frac{1}{3}}+\left(\frac{1}{3}\right)\sqrt{3\cdot\frac{1}{3}}+\left(\frac{1}{3}\right)\sqrt{3\cdot\frac{1}{3}}=\frac{1}{3}(1)+\frac{1}{3}(1)+\frac{1}{3}(1)=1$$

$$(x,y,z)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\begin{cases} x,y,z>0 \\ x+y+z=1 \\ x\sqrt{3y}+y\sqrt{3z}+z\sqrt{3x}=1 \end{cases}$$

$$x\sqrt{3y}=\sqrt{x}\cdot\sqrt{3xy}\le\frac{x+3xy}{2}$$

$$y\sqrt{3z}=\sqrt{y}\cdot\sqrt{3yz}\le\frac{y+3yz}{2}$$

$$z\sqrt{3x}=\sqrt{z}\cdot\sqrt{3zx}\le\frac{z+3zx}{2}$$

$$x\sqrt{3y}+y\sqrt{3z}+z\sqrt{3x}\le\frac{x+3xy}{2}+\frac{y+3yz}{2}+\frac{z+3zx}{2}$$

$$1\le\frac{(x+y+z)+3(xy+yz+zx)}{2}$$

$$x+y+z=1\Rightarrow 1\le\frac{1+3(xy+yz+zx)}{2}$$

$$\Leftrightarrow 2\le 1+3(xy+yz+zx)\Leftrightarrow 2-1\le 3(xy+yz+zx)\Leftrightarrow$$

$$\Leftrightarrow 3(xy+yz+zx)\ge 1 \quad \textbf{(i)}$$

$$(x-y)^2+(y-z)^2+(z-x)^2\ge 0$$

$$\Leftrightarrow x^2-2xy+y^2+y^2-2yz+z^2+z^2-2zx+x^2\ge 0$$

$$\Leftrightarrow 2(x^2+y^2+z^2)-2(xy+yz+zx)\ge 0\Leftrightarrow x^2+y^2+z^2\ge xy+y+zx$$

$$\Leftrightarrow (x^2+y^2+z^2)+2(xy+yz+zx)\ge(xy+yz+zx)+2(xy+yz+zx)$$

$$\Leftrightarrow (x+y+z)^2\ge 3(xy+yz+zx)$$

$$x+y+z=1\Rightarrow 1^2\ge 3(xy+yz+zx)\Leftrightarrow 3(xy+yz+zx)\le 1 \quad \textbf{(ii)}$$

$$\begin{cases} 3(xy+yz+zx)\ge 1 \\ 3(xy+yz+zx)\le 1 \end{cases}\Rightarrow 3(xy+yz+zx)=1$$

$$\Rightarrow \begin{cases} \sqrt{x}=\sqrt{3xy} \\ \sqrt{y}=\sqrt{3yz} \\ \sqrt{z}=\sqrt{3zx} \end{cases} \Leftrightarrow \begin{cases} x=3xy \\ y=3yz \\ z=3zx \end{cases} \Leftrightarrow \begin{cases} x(1-3y)=0 \\ y(1-3z)=0 \\ z(1-3x)=0 \end{cases}$$

$$x,y,z>0 \Rightarrow \begin{cases} 1-3y=0 \\ 1-3z=0 \\ 1-3x=0 \end{cases} \Leftrightarrow \begin{cases} 3y=1 \\ 3z=1 \\ 3x=1 \end{cases} \Leftrightarrow x=y=z=\frac{1}{3}$$

$$x+y+z=\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1$$

$$\left(\frac{1}{3}\right)\sqrt{3\cdot\frac{1}{3}}+\left(\frac{1}{3}\right)\sqrt{3\cdot\frac{1}{3}}+\left(\frac{1}{3}\right)\sqrt{3\cdot\frac{1}{3}}=\frac{1}{3}(1)+\frac{1}{3}(1)+\frac{1}{3}(1)=1$$

$$(x,y,z)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$$

Solution 5 by Hikmat Mammadov-Azerbaijan

$$\begin{cases} x,y,z>0 \\ x+y+z=1 \\ x\sqrt{3y}+y\sqrt{3z}+z\sqrt{3x}=1 \end{cases}$$

$$x\sqrt{3y}+y\sqrt{3z}+z\sqrt{3x}=1 \Leftrightarrow \sqrt{3}\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)=1$$

$$\Leftrightarrow x\sqrt{y}+y\sqrt{z}+z\sqrt{x}=\frac{1}{\sqrt{3}}$$

$$\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)^2=\left(\sqrt{x}\cdot\sqrt{xy}+\sqrt{y}\cdot\sqrt{yz}+\sqrt{z}\cdot\sqrt{zx}\right)^2$$

$$\leq\left[\left(\sqrt{x}\right)^2+\left(\sqrt{y}\right)^2+\left(\sqrt{z}\right)^2\right]\left[\left(\sqrt{xy}\right)^2+\left(\sqrt{yz}\right)^2+\left(\sqrt{zx}\right)^2\right]$$

$$=(x+y+z)(xy+yz+zx)=1\cdot(xy+yz+zx)=xy+yz+zx$$

$$(x-y)^2+(y-z)^2+(z-x)^2\geq 0$$

$$\Leftrightarrow x^2-2xy+y^2+y^2-2yz+z^2+z^2-2zx+x^2\geq 0 \Leftrightarrow$$

$$\Leftrightarrow 2(x^2+y^2+z^2)-2(xy+yz+zx)\geq 0$$

$$\Leftrightarrow x^2+y^2+z^2\geq xy+yz+zx \Leftrightarrow$$

$$\Leftrightarrow (x^2+y^2+z^2)+2(xy+yz+zx)\geq 3(xy+yz+zx)$$

$$\Leftrightarrow (x+y+z)^2\geq 3(xy+yz+zx)\Rightarrow 1^2\geq 3(xy+yz+zx)\Leftrightarrow xy+yz+zx\leq\frac{1}{3}$$

$$\frac{1}{3}=\left(\frac{1}{\sqrt{3}}\right)^2=\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)^2\leq xy+yz+zx\leq\frac{1}{3}$$

$$\Rightarrow xy+yz+zx=\frac{1}{3}\wedge\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)^2=xy+yz+zx$$

$$\Rightarrow (x+y+z)^2=1=3\left(\frac{1}{3}\right)=3(xy+yz+zx)$$

$$\Leftrightarrow x^2+y^2+z^2+2xy+2yz+2zx=3xy+3yz+3zx$$

$$\Leftrightarrow x^2+y^2+z^2-xy-yz-zx=0\Leftrightarrow$$

$$\Leftrightarrow 2(x^2+y^2+z^2-xy-yz-zx)=0$$

$$\Leftrightarrow (x-y)^2+(y-z)^2+(z-x)^2=0\Leftrightarrow$$

$$\Leftrightarrow x-y=0\wedge y-z=0\wedge z-x=0\Leftrightarrow x=y=z$$

$$x+y+z=1\wedge x=y=z\Rightarrow 3x=1\Leftrightarrow x=\frac{1}{3}\Rightarrow x=y=z=\frac{1}{3}$$

$$\left(\frac{1}{3}\right)\sqrt{3\cdot\frac{1}{3}}+\left(\frac{1}{3}\right)\sqrt{3\cdot\frac{1}{3}}+\left(\frac{1}{3}\right)\sqrt{3\cdot\frac{1}{3}}=\frac{1}{3}(1)+\frac{1}{3}(1)+\frac{1}{3}(1)=1$$

$$(x,y,z)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$$