ROMANIAN MATHEMATICAL MAGAZINE

JP.644 Solve for real numbers:

$$\sqrt{x^2+(4-x)^2}+\sqrt{y^2+(8-y)^2}=6\sqrt{2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\sqrt{\frac{x^2+(4-x)^2}{2}} \overset{QM-AM}{\geq} \frac{x+(4-x)}{2}=2$$

$$\sqrt{x^2+(4-x)^2}\geq 2\sqrt{2} \qquad (1)$$

$$\sqrt{\frac{y^2+(8-y)^2}{2}} \overset{QM-AM}{\geq} \frac{y+(8-y)}{2}=\frac{8}{2}=4$$

$$\sqrt{y^2+(8-y)^2}\geq 4\sqrt{2} \qquad (2)$$

By adding (1);(2):

$$LHS\geq 2\sqrt{2}+4\sqrt{2}=6\sqrt{2}$$

Equality holds for: $x=2; y=4$.

Solution 2 by Marin Chirciu-Romania

$$LHS=\sqrt{x^2+(4-x)^2}\sqrt{y^2+(8-y^2)} \overset{Minkowski}{\geq} \sqrt{(x+y)^2+(4-x+8-y)^2}=$$

$$=\sqrt{(x+y)^2+(12-x-y)^2} \overset{SOS}{\geq} 6\sqrt{2}=RHS,$$

where $\sqrt{(x+y)^2+(12-x-y)^2} \overset{SOS}{\geq} 6\sqrt{(x+y-6)^2}\geq 0$, with equal for

$\frac{x}{x-4}=\frac{y}{8-y}$ and $x+y=6 \Leftrightarrow (x,y)=(2,4)$.

We deduce that $(x,y)=(2,4)$ is the unique solution of the equation.

Remark.

Let be $a,b>0$. Solve for real numbers

$$\sqrt{x^2+(2a-x)^2}\sqrt{y^2+(2b-y)^2}=(a+b)\sqrt{2}$$

Marin Chirciu

Solution

$$LHS=\sqrt{x^2+(2a-x)^2}\sqrt{y^2+(2b-y)^2} \overset{Minkowski}{\geq} \sqrt{(x+y)^2+(2a-x+2b-y)^2}=$$

$$= \sqrt{(x+y)^2 + (2a+2b-x-y)^2} \overset{SOS}{\geq} (a+b)\sqrt{2} = RHS,$$

where $\sqrt{(x+y)^2 + (2a+2b-x-y)^2} \overset{SOS}{\geq} (a+b)\sqrt{2} \Leftrightarrow$

$$\Leftrightarrow (x+y-a-b)^2 \geq 0, \text{ with equal for}$$

$$\frac{x}{2a-x} = \frac{y}{2b-y} \text{ and } x+y = a+b \Leftrightarrow (x,y) = (a,b).$$

We deduce that $(x,y) = (a,b)$ is the unique solution of the equation.

Note.

For $(a,b) = (2,4)$ we obtain Problem JP.641 from RMM 43, Winter 2026.

Solution 3 by Jenish Rijal-Bardiya-Nepal

Here,

$$\sqrt{x^2 + (4-x)^2} + \sqrt{y^2 + (8-y)^2} = \sqrt{2x^2 - 8x + 16} + \sqrt{2y^2 - 16y + 64} =$$

$$= \sqrt{2(x-2)^2 + 8} + \sqrt{2(y-4)^2 + 32} \geq \sqrt{8} + \sqrt{32} = 6\sqrt{2}$$

Equality holds for $x = 2$ and $y = 4$.

Hence, $(x,y) = (2,4)$ is the unique solution.