ROMANIAN MATHEMATICAL MAGAZINE

JP.643 Let $\lambda > 0$ fixed. If $x, y, z > 0, x^2 + y^2 + z^2 = \lambda$ then find the maximum value of

$$P = \frac{x}{\lambda - yz} + \frac{y}{\lambda - zx} + \frac{z}{\lambda - xy}$$

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Solution 1 by proposer

With the substitution $\left(\frac{x}{\sqrt{\lambda}}, \frac{y}{\sqrt{\lambda}}, \frac{z}{\sqrt{\lambda}}\right) = (a, b, c)$ the problem can be reformulated.

If $a, b, c > 0, a^2 + b^2 + c^2 = 1$ then find the maximum value of

$$P = \frac{a}{1 - bc} + \frac{b}{1 - ca} + \frac{c}{1 - ab}.$$

We prove: Lemma:

If $a, b, c > 0, a^2 + b^2 + c^2 = 1$ then:

$$\frac{a}{1 - bc} \leq \frac{1}{\sqrt{3}}\left[1 + \frac{1}{2}\left(\frac{a^2}{a^2 + b^2} + \frac{a^2}{a^2 + c^2}\right)\right]$$

Proof: We have:

$$\frac{1}{1 - bc} \overset{AGM}{\leq} \frac{a}{1 - \frac{b^2 + c^2}{2}} = \frac{a}{1 - \frac{1 - a^2}{2}} = \frac{2a}{a^2 + 1} = \frac{1}{\sqrt{3}} \cdot \frac{2\sqrt{3}a}{a^2 + 1} \overset{AGM}{\leq} \frac{1}{\sqrt{3}} \cdot \frac{1 + 3a^2}{a^2 + 1} =$$

$$= \frac{1}{\sqrt{3}}\left(1 + \frac{2a^2}{a^2 + 1}\right) = \frac{1}{\sqrt{3}}\left(1 + \frac{2a^2}{a^2 + (a^2 + a^2 + a^2)}\right) =$$

$$= \frac{1}{\sqrt{3}}\left(1 + \frac{2a^2}{(a^2 + b^2) + (a^2 + c^2)}\right) = \frac{1}{\sqrt{3}}\left[1 + \frac{1}{2}\left(\frac{a^2}{a^2 + b^2} + \frac{a^2}{a^2 + c^2}\right)\right].$$

Equality holds if and only if $a = b = c = \frac{1}{\sqrt{3}}$.

Let's get back to the main problem. Using the Lemma, we obtain:

$$P = \sum \frac{a}{1 - bc} \leq \sum \frac{1}{\sqrt{3}}\left[1 + \frac{1}{2}\left(\frac{a^2}{a^2 + b^2} + \frac{a^2}{a^2 + c^2}\right)\right] =$$

$$= \frac{1}{\sqrt{3}}\left(3 + \frac{1}{2}\sum\left(\frac{a^2}{a^2 + b^2} + \frac{a^2}{a^2 + c^2}\right)\right) =$$

$$= \frac{1}{\sqrt{3}}\left(3 + \frac{1}{2}\sum\left(\frac{a^2}{a^2 + b^2} + \frac{b^2}{b^2 + a^2}\right)\right) = \frac{1}{\sqrt{3}}\left(3 + \frac{1}{2}\sum 1\right) = \frac{1}{\sqrt{3}}\left(3 + \frac{1}{2} \cdot 3\right) = \frac{3\sqrt{3}}{2}.$$

We deduce that $\max P = \frac{3\sqrt{3}}{2}$ and the maximum attained for $(a,b,c) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

Returning to the initial problem, we deduce that $\max P = \frac{3\sqrt{3}}{2}$ and the attained maximum for

$$(x,y,z) = \left(\sqrt{\frac{\lambda}{3}}, \sqrt{\frac{\lambda}{3}}, \sqrt{\frac{\lambda}{3}}\right).$$

Solution 2 by Jenish Rijal-Bardiya-Nepal

$$\text{Here, } P = \sum_{cyc} \frac{x}{\lambda - yz} \overset{\text{AM-GM}}{\leq} \sum_{cyc} \frac{x}{\lambda - \left(\frac{y^2+z^2}{2}\right)} = \sum_{cyc} \frac{2x}{2x^2+y^2+z^2} = \sum_{cyc} \frac{2x}{\lambda + x^2}$$

$$\text{Lemma: For any } t > 0, \quad \frac{2t}{1+t^2} \leq \frac{3t+\sqrt{3}}{4}$$

$$\text{Proof: } 3t^3 + \sqrt{3}t^2 + \sqrt{3} = \frac{15t^3}{5} + \frac{5\sqrt{3}t^2}{5} + 3 \cdot \frac{1}{\sqrt{3}} \overset{\text{Weighted AM-GM}}{\geq} 5 \cdot \sqrt[5]{\frac{15t^3}{5} \cdot \frac{5\sqrt{3}t^2}{5} \cdot \left(\frac{1}{\sqrt{3}}\right)^3} = 5t$$

$$\text{Thus, } \left(3t+\sqrt{3}\right)\left(1+t^2\right) = 3t^3 + \sqrt{3}t^2 + \sqrt{3} + 3t \geq 5t + 3t = 8t \ \therefore \frac{2t}{1+t^2} \leq \frac{3t+\sqrt{3}}{4}$$

$$\text{With } t = \frac{x}{\sqrt{\lambda}}, \text{ we get: } \frac{2x}{\lambda + x^2} \leq \frac{3x+\sqrt{3\lambda}}{4\lambda}$$

$$\text{Hence, } \quad P \leq \sum_{cyc} \frac{2x}{\lambda + x^2} \leq \sum_{cyc} \frac{3x+\sqrt{3\lambda}}{4\lambda} = \frac{3(x+y+z)+3\sqrt{3\lambda}}{4\lambda} \overset{(\sum x)^2 \leq 3(\sum x^2)}{\leq}$$

$$\leq \frac{3\sqrt{3(x^2+y^2+z^2)}+3\sqrt{3\lambda}}{4\lambda} = \frac{3\sqrt{3(x^2+y^2+z^2)}+3\sqrt{3\lambda}}{4\lambda} = \frac{3\sqrt{3(\lambda)}+3\sqrt{3\lambda}}{4\lambda} = \frac{3\sqrt{3}}{2\sqrt{\lambda}}$$

Equality holds if and only if $a = b = c = \sqrt{\frac{\lambda}{3}}$. **Remark.**

Alternatively, the inequality can be proved directly using *Jensen's* inequality:

$$P \leq \sum_{cyc} \frac{2x}{\lambda + x^2} \overset{\text{Jensen}}{\leq} 3f\left(\frac{x+y+z}{3}\right) \overset{(\sum x)^2 \leq 3(\sum x^2)}{\leq} 3f\left(\frac{\sqrt{3(x^2+y^2+z^2)}}{3}\right) = 3f\left(\sqrt{\frac{\lambda}{3}}\right) = \frac{3\sqrt{3}}{2\sqrt{\lambda}}$$

$$\text{where } f(t) = \frac{2t}{\lambda + t^2}, \text{ which is increasing and concave on } (0, \sqrt{\lambda}\,].$$