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JP.642 If $a,b,c>0,\sum a=3$ and $n\in\mathbb{N}$ then:

$$\sum\left(\frac{a^3}{a^2+ab+b^2}\right)^n\geq\frac{3}{3^n}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

For $n=0$ we obtain equality $3=3$.

For $n=1$ see Lemma.

For $n\geq 2$ we use Holder's inequality.

$$LHS=\sum\left(\frac{a^3}{a^2+ab+b^2}\right)^n\overset{Holder}{\geq}\frac{\left(\sum\frac{a^3}{a^2+ab+b^2}\right)^n}{3^{n-1}}\overset{Lemma}{\geq}\frac{1^n}{3^{n-1}}=\frac{3}{3^n}=RHS.$$

Lemma.

If $a,b,c>0,\sum a=3$ then:

$$\sum\frac{a^3}{a^2+ab+b^2}\geq 1.$$

Proof

If $a,b>0$ then:

$$\frac{a^3}{a^2+ab+b^2}\geq\frac{2a-b}{3}$$

$$\frac{a^3}{a^2+ab+b^2}\geq\frac{2a-b}{3}\Leftrightarrow 3a^3\geq(2a-b)(a^2+ab+b^2)\Leftrightarrow(a-b)^2(a+b)\geq 0$$

Let's get back to the main problem.

$$LHS=\sum\frac{a^3}{a^2+ab+b^2}\geq\sum\frac{2a-b}{3}=\frac{1}{3}\sum a=\frac{1}{3}\cdot 3=1=RHS.$$

Equality holds if and only if $a=b=c=1$.

Solution 2 by Jose Luis Diaz Barrero-Spain

For $a,b>0$ we have

$$\frac{a^3}{a^2+ab+b^2}-\frac{2a-b}{3}=\frac{(a+b)(a-b)^2}{3(a^2+ab+b^2)}\geq 0.$$

Hence, adding cyclically and using $a+b+c=3$, we have

$$\sum_{cyc}\frac{a^3}{a^2+ab+b^2}\geq\frac{a+b+c}{3}=1.$$

Write $x=\frac{a^3}{a^2+ab+b^2}$, $y=\frac{b^3}{b^2+bc+c^2}$ and $z=\frac{c^3}{c^2+ca+a^2}$. All three are positive. For $n\geq 1$ the function $t\to t^n$ is convex on $(0,\infty)$, so by Jensen's inequality, we have

$$x^n+y^n+z^n\geq 3\left(\frac{x+y+z}{3}\right)^n=\frac{S^n}{3^{n-1}}\geq\frac{1}{3^{n-1}}=\frac{3}{3^n},$$

where $S=x+y+z$. Equality holds if and only if $a=b=c=1$.

Note. If $n=0$ the inequality trivially holds, as can be immediately checked.

Solution 3 by Jenish Rijal-Bardiya-Nepal

For $n=0$, the inequality becomes trivial. For $n=1$, the inequality becomes:

$$\sum_{cyc}\frac{a^3}{a^2+ab+b^2}\geq 1$$

Here,

$$\sum_{cyc}\frac{a^3}{a^2+ab+b^2}=\sum_{cyc}\frac{(a^2)^2}{a(a^2+ab+b^2)}\overset{\text{Bergström}}{\geq}\frac{(\sum a^2)^2}{\sum a(a^2+ab+b^2)}=\frac{(\sum a^2)^2}{(\sum a)(\sum a^2)}=$$

$$=\frac{\sum a^2}{\sum a}\overset{3\sum a^2\geq(\sum a)^2}{\geq}\frac{\sum a}{3}=\frac{3}{3}=1$$

For $n\geq 2$,

$$\sum_{cyc}\left(\frac{a^3}{a^2+ab+b^2}\right)^n\overset{\text{Power Mean}}{\geq}3\cdot\left(\frac{1}{3}\cdot\sum_{cyc}\frac{a^3}{a^2+ab+b^2}\right)^n\geq 3\cdot\left(\frac{1}{3}\cdot 1\right)^n=\frac{3}{3^n}$$

$$Equality\ holds\ if\ and\ only\ if\quad a=b=c=1.$$