ROMANIAN MATHEMATICAL MAGAZINE

JP.641 Solve for real numbers:

$$4\sqrt{x-4}+10\sqrt{y-25}+14\sqrt{z-49}=x+y+z$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$4\sqrt{x-4}+10\sqrt{y-25}+14\sqrt{z-49}=$$

$$=2\left(2\sqrt{x-4}+5\sqrt{y-25}+7\sqrt{z-49}\right)=$$

$$=2\left(\sqrt{4(x-4)}+\sqrt{25(y-25)}+\sqrt{49(z-49)}\right)\overset{AM-GM}{\leq}$$

$$\leq 2\cdot\left(\frac{4+x-4}{2}+\frac{25+y-25}{2}+\frac{49+z-49}{2}\right)=$$

$$=2\frac{x+y+z}{2}=x+y+z$$

Equality holds for:

$$\begin{cases}4=x-4\\25=y-25\\49=z-49\end{cases}\Leftrightarrow\begin{cases}x=8\\y=50\\z=98\end{cases}$$

Solution 2 by Marin Chirciu-Romania

$$LHS=4\sqrt{x-4}+10\sqrt{y-25}+14\sqrt{z-49}\overset{CBS}{\leq}$$

$$\leq\sqrt{(4^2+10^2+14^2)(x-4+y-25+z-49)}=$$

$$=\sqrt{312(x+y+z-78)}\overset{SOS}{\leq}x+y+z=RHS,$$

where $\sqrt{312(x+y+z-78)}\overset{SOS}{\leq}x+y+z\Leftrightarrow(x+y+z-156)^2\geq 0$, with equal for

$\frac{\sqrt{x-4}}{4^2}=\frac{\sqrt{y-25}}{10^2}=\frac{\sqrt{z-49}}{14^2}$ and $x+y+z=156\Leftrightarrow(x,y,z)=(8,50,98)$.

We deduce that $(x,y,z)=(8,50,98)$ is the unique solution of the equation.

Remark.

Let be $a,b,c>0$ fixed. Solve for real numbers:

$$2a\sqrt{x-a^2}+2b\sqrt{y-b^2}+2c\sqrt{z-c^2}=x+y+z.$$

Marin Chirciu

Solution

$$LHS = 2a\sqrt{x-a^2} + 2b\sqrt{y-b^2} + 2c\sqrt{z-c^2} \overset{CBS}{\leq}$$

$$\leq \sqrt{(4a^2+4b^2+4c^2)(x-a^2+y-b^2+z-c^2)} =$$

$$= \sqrt{(4a^2+4b^2+4c^2)(x+y+z-a^2-b^2-c^2)} \overset{SOS}{\leq} x+y+z = RHS,$$

where $\sqrt{(4a^2+4b^2+4c^2)(x+y+z-a^2-b^2-c^2)} \overset{SOS}{\leq} x+y+z \Leftrightarrow$

$$\Leftrightarrow \left(x+y+z-2(a^2+b^2+c^2)\right)^2 \geq 0, \text{ with equal for:}$$

$\frac{\sqrt{x-a^2}}{4a^2} = \frac{\sqrt{y-b^2}}{4b^2} = \frac{\sqrt{z-c^2}}{4c}$ and $x+y+z = 2(a^2+b^2+c^2) \Leftrightarrow (x,y,z) = (2a^2, 2b^2, 2c^2)$.

We deduce that $(x,y,z) = (2a^2, 2b^2, 2c^2)$ is the unique solution of the equation.

Note.

For $(a,b,c) = (2,5,7)$ we obtain Problem JP.641 from RMM 43, Winter 2026.

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\begin{cases} x-4 \geq 0 \\ y-25 \geq 0 \\ z-49 \geq 0 \end{cases} \Leftrightarrow \begin{cases} x \geq 4 \\ y \geq 25 \\ z \geq 49 \end{cases}$$

$$\forall t > 0, \int_0^{\sqrt{a-t^2}} 2(t-u)\,du = 2\left[tu - \frac{u^2}{2}\right]_0^{\sqrt{a-t^2}} = [2tu-u^2]_0^{\sqrt{a-t^2}}$$

$$= 2t\sqrt{a-t^2} - \left(\sqrt{a-t^2}\right)^2 - (0-0) = 2t\sqrt{a-t^2} - (a-t^2) = 2t\sqrt{a-t^2} - a + t^2$$

$$\left(t-\sqrt{a-t^2}\right)^2 \geq 0 \Leftrightarrow t^2 - 2t\sqrt{a-t^2} + \left(\sqrt{a-t^2}\right)^2 \geq 0$$

$$\Leftrightarrow t^2 - 2t\sqrt{a-t^2} + a - t^2 \geq 0 \Leftrightarrow a - 2t\sqrt{a-t^2} \geq 0 \Leftrightarrow 2t\sqrt{a-t^2} \leq a$$

$$\int_0^{\sqrt{a-t^2}} 2(t-u)\,du = 2t\sqrt{a-t^2} - a + t^2 \leq a - a + t^2 = t^2$$

$$t_1 = 2\ (t_1^2 = 4) \Rightarrow \int_0^{\sqrt{x-4}} 2(2-u)du = 4\sqrt{x-4} - (x-4) \leq 4$$

$$\Rightarrow 4\sqrt{x-4} - x + 4 \leq 4 \Rightarrow 4\sqrt{x-4} \leq x$$

Equality $\Leftrightarrow 2 - \sqrt{x-4} = 0 \Leftrightarrow \sqrt{x-4} = 2 \Leftrightarrow x-4 = 4 \Leftrightarrow x = 8$

$$t_2 = 5\ (t_2^2 = 25) \Rightarrow \int_0^{\sqrt{y-25}} 2(5-u)\,du = 10\sqrt{y-25} - (y-25) \leq 25$$

$$\Rightarrow 10\sqrt{y-25} - y + 25 \leq 25 \Rightarrow 10\sqrt{y-25} \leq y$$

Equality $\Leftrightarrow 5-\sqrt{y-25}=0 \Leftrightarrow \sqrt{y-25}=5 \Leftrightarrow y-25=25 \Leftrightarrow y=50$

$$t_3=7\ (t_3^2=49) \Rightarrow \int_0^{\sqrt{z-49}} 2(7-u)\,du = 14\sqrt{z-49}-(z-49) \le 49$$

$$\Rightarrow 14\sqrt{z-49}-z+49 \le 49 \Rightarrow 14\sqrt{z-49} \le z$$

Equality $\Leftrightarrow 7-\sqrt{z-49}=0 \Leftrightarrow \sqrt{z-49}=7 \Leftrightarrow z-49=49 \Leftrightarrow z=98$

$$\sum \Rightarrow 4\sqrt{x-4}+10\sqrt{y-25}+14\sqrt{z-49} \le x+y+z$$

$$\begin{cases} 4\sqrt{x-4}=x \\ 10\sqrt{y-25}=y \\ 14\sqrt{z-49}=z \end{cases} \Leftrightarrow \begin{cases} \sqrt{x-4}=2 \\ \sqrt{y-25}=5 \\ \sqrt{z-49}=7 \end{cases} \Leftrightarrow \begin{cases} x-4=4 \\ y-25=25 \\ z-49=49 \end{cases} \Leftrightarrow \begin{cases} x=8 \\ y=50 \\ z=98 \end{cases}$$

$$(x,y,z)=(8,50,98)$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\begin{cases} x-4\ge 0 \\ y-25\ge 0 \\ z-49\ge 0 \end{cases} \Leftrightarrow \begin{cases} x\ge 4 \\ y\ge 25 \\ z\ge 49 \end{cases}$$

$$f(t)=\sqrt{t},\ \ t>0$$

$$f'(t)=\frac{1}{2\sqrt{t}},\ \ f''(t)=-\frac{1}{4t^{\frac{3}{2}}}<0.\ (\forall t>0) \Rightarrow f(t) \text{ is strictly concave}$$

Jensen's Inequality: $\lambda_1 f(t_1)+\lambda_2 f(t_2)+\lambda_3 f(t_3) \le f(\lambda_1 t_1+\lambda_2 t_2+\lambda_3 t_3)$

$$\lambda_1=\frac{4}{78},\lambda_2=\frac{25}{78},\lambda_3=\frac{49}{78}\ \ \left(\lambda_1+\lambda_2+\lambda_3=\frac{4+25+49}{78}=1\right)$$

$$t_1=\frac{x-4}{4},t_2=\frac{y-25}{25},t_3=\frac{z-49}{49}$$

$$\frac{4}{78}\sqrt{\frac{x-4}{4}}+\frac{25}{78}\sqrt{\frac{y-25}{25}}+\frac{49}{78}\sqrt{\frac{z-49}{49}} \le$$

$$\le \sqrt{\frac{4}{78}\cdot\frac{x-4}{4}+\frac{25}{78}\cdot\frac{y-25}{25}+\frac{49}{78}\cdot\frac{z-49}{49}}$$

$$\frac{4}{78}\cdot\frac{\sqrt{x-4}}{2}+\frac{25}{78}\cdot\frac{\sqrt{y-25}}{5}+\frac{49}{78}\cdot\frac{\sqrt{z-49}}{7} \le \sqrt{\frac{(x-4)+(y-25)+(z-49)}{78}}$$

$$\frac{2\sqrt{x-4}+5\sqrt{y-25}+7\sqrt{z-49}}{78} \le \sqrt{\frac{x+y+z-78}{78}}$$

$$2\sqrt{x-4}+5\sqrt{y-25}+7\sqrt{z-49}\le 78\sqrt{\frac{x+y+z-78}{78}}$$

$$2\left(2\sqrt{x-4}+5\sqrt{y-25}+7\sqrt{z-49}\right)\le 2\cdot 78\sqrt{\frac{x+y+z-78}{78}}$$

$$4\sqrt{x-4}+10\sqrt{y-25}+14\sqrt{z-49}\le 156\sqrt{\frac{x+y+z-78}{78}}$$

$$x+y+z\le 156\sqrt{\frac{x+y+z-78}{78}}\Rightarrow (x+y+z)^2\le 156^2\cdot\frac{x+y+z-78}{78}$$

$$(x+y+z)^2\le 312(x+y+z-78)\Leftrightarrow (x+y+z-156)^2\le 0\Rightarrow$$

$$\Rightarrow x+y+z=156$$

Equality Condition $\Leftrightarrow t_1=t_2=t_3$

$$\Leftrightarrow\frac{x-4}{4}=\frac{y-25}{25}=\frac{z-49}{49}=k$$

$$\Leftrightarrow\begin{cases}x-4=4k\\y-25=25k\\z-49=49k\end{cases}\Leftrightarrow\begin{cases}x=4k+4\\y=25k+25\\z=49k+49\end{cases}$$

$$x+y+z=(4k+4)+(25k+25)+(49k+49)=78k+78=156\Leftrightarrow$$

$$\Leftrightarrow 78k=78\Leftrightarrow k=1$$

$$\Rightarrow\begin{cases}x=4(1)+4\\y=25(1)+25\\z=49(1)+49\end{cases}\Leftrightarrow\begin{cases}x=4+4\\y=25+25\\z=49+49\end{cases}\Leftrightarrow\begin{cases}x=8\\y=50\\z=98\end{cases}$$

$$(x,y,z)=(8,50,98)$$

Solution 5 by Hikmat Mammadov-Azerbaijan

$$\begin{cases}x-4\ge 0\\y-25\ge 0\\z-49\ge 0\end{cases}\Leftrightarrow\begin{cases}x\ge 4\\y\ge 25\\z\ge 49\end{cases}$$

$$u=\left(\sqrt{x-4},\sqrt{y-25},\sqrt{z-49}\right),\ \ v=(2,5,7)$$

$$u\cdot v=2\sqrt{x-4}+5\sqrt{y-25}+7\sqrt{z-49}$$

$$\|u\|^2=\left(\sqrt{x-4}\right)^2+\left(\sqrt{y-25}\right)^2+\left(\sqrt{z-49}\right)^2=(x-4)+(y-25)+(z-49)=$$

$$=x+y+z-78$$

$$\|v\|^2=2^2+5^2+7^2=4+25+49=78$$

ROMANIAN MATHEMATICAL MAGAZINE

$$(u \cdot v)^2 \leq \|u\|^2 \cdot \|v\|^2$$

$$\left(2\sqrt{x-4}+5\sqrt{y-25}+7\sqrt{z-49}\right)^2 \leq (x+y+z-78)\cdot 78$$

$$4\left(2\sqrt{x-4}+5\sqrt{y-25}+7\sqrt{z-49}\right)^2 \leq 4\cdot 78\cdot (x+y+z-78)$$

$$\left(4\sqrt{x-4}+10\sqrt{y-25}+14\sqrt{z-49}\right)^2 \leq 312(x+y+z-78)$$

$$(x+y+z)^2 \leq 312(x+y+z)-312\cdot 78$$

$$(x+y+z)^2 \leq 312(x+y+z)-24336$$

$$(x+y+z)^2-312(x+y+z)+24336 \leq 0$$

$$(x+y+z)^2-2\cdot 156\cdot (x+y+z)+156^2 \leq 0$$

$$(x+y+z-156)^2 \leq 0$$

$$\forall S \in \mathbb{R}_1 (S-156)^2 \geq 0 \Rightarrow (x+y+z-156)^2 = 0 \Rightarrow x+y+z=156$$

Equality Condition $\Leftrightarrow \frac{\sqrt{x-4}}{2}=\frac{\sqrt{y-25}}{5}=\frac{\sqrt{z-49}}{7}=k$

$$\Leftrightarrow \begin{cases} \sqrt{x-4}=2k \\ \sqrt{y-25}=5k \\ \sqrt{z-49}=7k \end{cases} \Leftrightarrow \begin{cases} x-4=4k^2 \\ y-25=25k^2 \\ z-49=49k^2 \end{cases} \Leftrightarrow \begin{cases} x=4k^2+4 \\ y=25k^2+25 \\ z=49k^2+49 \end{cases}$$

$$x+y+z=(4k^2+4)+(25k^2+25)+(49k^2+49)=78k^2+78$$

$$78k^2+78=156 \Leftrightarrow 78k^2=78 \Leftrightarrow k^2=1 \Leftrightarrow k=1 \ \ (k>0)$$

$$\Rightarrow \begin{cases} x=4(1)^2+4 \\ y=25(1)^2+25 \\ z=49(1)^2+49 \end{cases} \Leftrightarrow \begin{cases} x=4+4 \\ y=25+25 \\ z=49+49 \end{cases} \Leftrightarrow \begin{cases} x=8 \\ y=50 \\ z=98 \end{cases}$$

$$(x,y,z)=(8,50,98)$$