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JP.640 If $x, y, z > 0, x + y + z = \frac{\pi}{2}$ then:

$$\sum \left(\frac{\cos(y-z)}{\sin x}\right)^n \geq 3 \cdot 2^n$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

For $n = 0$ we obtain equality $3 = 3$. For $n = 1$ see Lemma.

For $n \geq 2$ we use Holder's inequality.

$$LHS = \sum \left(\frac{\cos(y-z)}{\sin x}\right)^n \overset{Holder}{\geq} \frac{\left(\sum \frac{\cos(y-z)}{\sin x}\right)^n}{3^{n-1}} \overset{Lemma}{\geq} \frac{6^n}{3^{n-1}} = 3 \cdot 2^n = RHS.$$

Lemma: If $x, y, z > 0, x + y + z = \frac{\pi}{2}$ then: $\sum \frac{\cos(y-z)}{\sin x} \geq 6$.

Proof: If $x, y, z > 0, x + y + z = \frac{\pi}{2}$ then: $\sum \tan x \tan y = 1$.

$$x + y + z = \frac{\pi}{2} \Leftrightarrow x + y = \frac{\pi}{2} - z \Leftrightarrow \tan(x+y) = \tan\left(\frac{\pi}{2} - z\right) \Leftrightarrow \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$= \cot z$$

$$\Leftrightarrow \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{1}{\tan z} \Leftrightarrow \tan z (\tan x + \tan y) = 1 - \tan x \tan y$$

$$\Leftrightarrow \sum \tan x \tan y = 1$$

$$\frac{\cos(y-z)}{\sin x} = \frac{\cos(y-z)}{\sin\left(\frac{\pi}{2} - y - z\right)} = \frac{\cos(y-z)}{\cos(y+z)} = \frac{\cos y \cos z + \sin y \sin z}{\cos y \cos z - \sin y \sin z}$$

$$= \frac{1 + \tan y \tan z}{1 - \tan y \tan z} =$$

$$= 1 + \frac{2 \tan y \tan z}{1 - \tan y \tan z} = 1 + \frac{2 \tan y \tan z}{\tan x \tan y + \tan x \tan z}.$$

$$LHS = \sum \frac{\cos(y-z)}{\sin x} = \sum \left(1 + \frac{2 \tan y \tan z}{\tan x \tan y + \tan x \tan z}\right) =$$

$$= 3 + 2 \sum \frac{\tan y \tan z}{\tan x \tan y + \tan x \tan z} \overset{Nesbitt}{\geq} 3 + 2 \cdot \frac{3}{2} = 6 = RHS.$$

Equality holds if and only if $x = y = z = \frac{\pi}{6}$.

Solution 2 by Jose Luis Diaz Barrero-Spain

Since $x=\frac{\pi}{2}-(y+z)$, then we have $\sin x=\cos(y+z)>0$, because $0<y+z<\frac{\pi}{2}$.

Hence

$$\frac{\cos(y-z)}{\sin x}=\frac{\cos(y-z)}{\cos(y+z)}=\frac{\cos y\cos z+\sin y\sin z}{\cos y\cos z-\sin y\sin z}=\frac{1+\tan y\tan z}{1-\tan y\tan z}$$

Put $u=\tan y\tan z$, $v=\tan z\tan x$ and $w=\tan x\tan y$. These are positive numbers and, since $x+y+z=\frac{\pi}{2}$, it is well known that

$$\tan x\tan y+\tan y\tan z+\tan z\tan x=1,$$

that is, $u+v+w=1$. In particular $0<u,v,w<1$. Setting

$$A=\frac{1+u}{1-u},\quad B=\frac{1+v}{1-v},\quad C=\frac{1+w}{1-w},$$

we must prove that $A^n+B^n+C^n\geq 3\cdot 2^n$. Indeed, since $\frac{1+u}{1-u}=\frac{2}{1-u}-1$, the AM-HM inequality yields

$$A+B+C=2\left(\frac{1}{1-u}+\frac{1}{1-v}+\frac{1}{1-w}\right)-3\geq 2\cdot\frac{9}{(1-u)+(1-v)+(1-w)}-3$$
$$=2\cdot\frac{9}{2}-3=6.$$

Finally, by the power mean inequality, we have

$$A^n+B^n+C^n\geq 3\left(\frac{A+B+C}{3}\right)^n\geq 3\cdot 2^n,$$

as desired. Equality holds if and only if $u=v=w=\frac{1}{3}$, and substituting in the preceding we get equality when $x=y=z=\frac{\pi}{6}$.

Solution 3 by Jenish Rijal-Bardiya-Nepal

For $n=0$, the inequality is trivial and we get equality.

Let $a=\tan y\tan z$, $b=\tan z\tan x$, $c=\tan x\tan y$. Since $x+y+z=\frac{\pi}{2}$, we have: $a+b+c=1$.

Here, $$\frac{\cos(y-z)}{\sin x}=\frac{\cos(y-z)}{\cos(y+z)}=\frac{1+\tan y\tan z}{1-\tan y\tan z}=\frac{1+a}{1-a}=\frac{1+a}{b+c}\quad\text{(and analogs)}$$

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$$\sum_{cyc}\frac{\cos(y-z)}{\sin x}=\sum_{cyc}\frac{1+a}{b+c}=\sum_{cyc}\frac{a}{b+c}+\sum_{cyc}\frac{1}{b+c}\overset{\substack{\text{Nesbitt}\\+\\\text{Bergström}}}{\geq}$$

$$\geq\frac{3}{2}+\frac{(1+1+1)^2}{2(a+b+c)}=\frac{3}{2}+\frac{9}{2}=6$$

Finally for $n>0$: $\sum_{cyc}\left(\frac{\cos(y-z)}{\sin x}\right)^n\overset{\text{Power Mean}}{\geq}3\cdot\left(\frac{1}{3}\cdot\sum_{cyc}\frac{\cos(y-z)}{\sin x}\right)^n\geq 3\cdot\left(\frac{6}{3}\right)^n=3\cdot 2^n$

Equality for $n>0$ holds if and only if $x=y=z=\frac{\pi}{6}$.

Remark: For $0<n<1$, Power Mean is not valid. But for every $0<n<1$,

$f(t)=\left(\frac{1+t}{1-t}\right)^n;n>0$ is strictly convex. Hence the desired inequality

can be proven also (for $0<n<1$)via $Jensen's$ inequality.