ROMANIAN MATHEMATICAL MAGAZINE

JP.639 If $x,y,z>0, \sum x=1$ and $\lambda \leq 4$ then:

$$\lambda \sum x^3+3(\lambda+1)xyz \geq \frac{2\lambda+1}{9}$$

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Solution 1 by proposer

Using pqr – Method

We denote $p=\sum x=1, q=\sum xy, r=xyz$.

We have $\sum x^3=p^3-3pq+3r, p=1 \Leftrightarrow \sum x^3=1-3q+3r$.

We write the inequality:

$$\lambda(1-3q+3r)+3(\lambda+1)r \geq \frac{2\lambda+1}{9} \Leftrightarrow \frac{7\lambda-1}{9}-3\lambda q+3(2\lambda+1)r \geq 0,$$

which follows from Schur's inequality

$$p^3-4pq+9r \geq 0, p=1 \Leftrightarrow 1-4q+9r \geq 0 \Leftrightarrow r \geq \frac{4q-1}{9}$$

It suffices to show that:

$$\frac{7\lambda-1}{9}-3\lambda q+3(2\lambda+1)\cdot\frac{4q-1}{9} \geq 0 \Leftrightarrow 3(\lambda-4)q \leq 1 \Leftrightarrow 3q \leq 1,$$

see $1=p^2=(x+y+z)^2 \geq 3(xy+yz+zx)=3q$.

Equality holds if and only if $x=y=z=\frac{1}{3}$.

Solution 2 by Jenish Rijal-Bardiya-Nepal

$$\text{Let } \; p=x+y+z=1, \qquad q=xy+yz+zx, \qquad r=xyz.$$

$$p^3+9r \overset{\text{Schur's}}{\geq} 4pq \overset{p=1}{\Leftrightarrow} r \geq \frac{4q-1}{9}. \text{Here:}$$

$$x^3+y^3+z^3=p^3-3pq+3r=1-3q+3r.$$

Now,

$$\lambda(x^3+y^3+z^3)+3(\lambda+1)r=\lambda-3\lambda q+(6\lambda+3)r \geq$$

$$\geq \lambda-3\lambda q+(6\lambda+3)\frac{4q-1}{9}=\frac{2\lambda+1}{9}+\frac{(\lambda-4)(1-3q)}{9} \geq \frac{2\lambda+1}{9}$$

where the last inequality follows from

$3q = 3(xy + yz + zx) \leq (x + y + z)^2 = 1$ and $\lambda \geq 4$.

Remark: The condition $\lambda \geq 4$ is both necessary and sufficient (the printed condition $\lambda \leq 4$ is incorrect.) Indeed, take $x = y = \frac{1-t}{2}, z = t$ with $0 < t < 1$. As $t \to 0^+$, we obtain

$$\lambda(x^3 + y^3 + z^3) + 3(\lambda + 1)xyz \to \frac{\lambda}{4}.$$

$$\text{Thus, if } \lambda < 4, \text{ then } \frac{\lambda}{4} < \frac{2\lambda + 1}{9}$$

,so the proposed inequality fails for sufficiently small positive t.

Therefore, the exact condition is $\boxed{\lambda \geq 4.}$