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JP.638 If $a, b, c > 0$, prove that :

$$a + b + c \leq \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab}$$

Proposed by Neculai Stanciu – Romania

Solution 1 by proposer

By AM-GM $a^4 + b^4 + c^4 = \frac{a^4+b^4}{2} + \frac{b^4+c^4}{2} + \frac{c^4+a^4}{2} \geq a^2b^2 + b^2c^2 + c^2a^2 =$

$$= a^2\left(\frac{b^2+c^2}{2}\right) + b^2\left(\frac{c^2+a^2}{2}\right) + c^2\left(\frac{a^2+b^2}{2}\right) \geq a^2bc + b^2ca + c^2ab =$$

$= abc(a + b + c)$, q.e.d. Equality iff $a = b = c$.

Solution 2 by Marin Chirciu-Romania

$$RHS = \sum \frac{a^3}{bc} \overset{Holder}{\geq} \frac{(\sum a)^3}{3\sum bc} \overset{SOS}{\geq} \frac{(\sum a)^3}{(\sum a)^2} = \sum a = LHS$$

Equality holds if and only if $a = b = c$.

Remark.

If $a, b, c > 0$ and $n \in \mathbb{N}^*$ then

$$3\left(\frac{\sum a}{3}\right)^{2n-1} \leq \sum \frac{a^{2n+1}}{bc}.$$

Marin Chirciu

Solution.

$$RHS = \sum \frac{a^{2n+1}}{bc} \overset{Holder}{\geq} \frac{(\sum a)^{2n+1}}{3^{2n-1}\sum bc} \overset{SOS}{\geq} \frac{(\sum a)^{2n-1}}{3^{2n-1}\frac{1}{3}(\sum a)^2} =$$

$$= \frac{(\sum a)^{2n-1}}{3^{2n-2}} = 3\left(\frac{\sum a}{3}\right)^{2n-1} = LHS$$

Equality holds if and only if $a = b = c$.

Note.

For $n = 1$ we obtain Problem JP.638 from RMM 43.

Solution 3 by Jose Luis Diaz Barrero-Spain

Since

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} = \frac{a^4 + b^4 + c^4}{abc}$$

and $abc > 0$, the inequality is equivalent to

$$a^4 + b^4 + c^4 \geq abc(a + b + c)$$

Recall that for all real numbers u, v, w, it holds

$$u^2 + v^2 + w^2 - uv - vw - wu = \frac{1}{2}[(u - v)^2 + (v - w)^2 + (w - u)^2] \geq 0$$

Applying this with $(u, v, w) = (a^2, b^2, c^2)$ and then with $(u, v, w) = (ab, bc, ca)$, we obtain

$$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c),$$

as desired. Equality holds if and only if $a = b = c$.

Solution 4 by Himat Mammadov-Azerbaijan

$$\int_y^x (t - y)\, dt = \left[\frac{t^2}{2} - yt\right]_y^x = \left(\frac{x^2}{2} - yx\right) - \left(\frac{y^2}{2} - y^2\right) = \frac{x^2 - 2xy + y^2}{2} = \frac{(x - y)^2}{2} \geq 0$$

$$\int_y^x (t - y)\, dt \geq 0 \Rightarrow \frac{x^2 + y^2}{2} \geq xy \Rightarrow x^2 + y^2 \geq 2xy$$

$$\int_{bc}^{a^2} (t - bc)\, dt + \int_{ca}^{b^2} (t - ca)\, dt + \int_{ab}^{c^2} (t - ab)\, dt \geq 0$$

$$\frac{(a^2 - bc)^2 + (b^2 - ca)^2 + (c^2 - ab)^2}{2} \geq 0$$

$$(a^4 - 2a^2bc + b^2c^2) + (b^4 - 2ab^2c + c^2a^2) + (c^4 - 2abc^2 + a^2b^2) \geq 0$$

$$a^4 + b^4 + c^4 + a^2b^2 + b^2c^2 + c^2a^2 \geq 2abc(a + b + c)$$

$$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2$$

$$a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$$

$$a^4 + b^4 + c^4 + a^4 + b^4 + c^4 \geq 2 \cdot (a^2b^2 + b^2c^2 + c^2a^2) \geq 2abc(a + b + c)$$

$$a^4 + b^4 + c^4 \geq abc(a + b + c)$$

$$\frac{a^4 + b^4 + c^4}{abc} \geq \frac{abc(a + b + c)}{abc}$$

$$\frac{a^4}{abc} + \frac{b^4}{abc} + \frac{c^4}{abc} \geq a + b + c$$

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c$$

$$a + b + c \leq \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \Rightarrow Q.E.D.$$

Solution 5 by Himat Mammadov-Azerbaijan

$$\bar{u}=\left(\sqrt{\frac{a^3}{bc}},\sqrt{\frac{b^3}{ca}},\sqrt{\frac{c^3}{ab}}\right),\bar{v}=\left(\sqrt{abc},\sqrt{abc},\sqrt{abc}\right)$$

$$(\bar{u}\cdot\bar{v})^2\le|\bar{u}|^2\cdot|\bar{v}|^2$$

$$\bar{u}\cdot\bar{v}=\sqrt{\frac{a^3}{bc}}\cdot\sqrt{abc}+\sqrt{\frac{b^3}{ca}}\cdot\sqrt{abc}+\sqrt{\frac{c^3}{ab}}\cdot\sqrt{abc}$$

$$\bar{u}\cdot\bar{v}=\sqrt{\frac{a^3\cdot abc}{bc}}+\sqrt{\frac{b^3\cdot abc}{ca}}+\sqrt{\frac{c^3\cdot abc}{ab}}=\sqrt{a^4}+\sqrt{b^4}+\sqrt{c^4}=a^2+b^2+c^2$$

$$|\bar{u}|^2=\left(\sqrt{\frac{a^3}{bc}}\right)^2+\left(\sqrt{\frac{b^3}{ca}}\right)^2+\left(\sqrt{\frac{c^3}{ab}}\right)^2=\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}$$

$$|\bar{v}|^2=\left(\sqrt{abc}\right)^2+\left(\sqrt{abc}\right)^2+\left(\sqrt{abc}\right)^2=abc+abc+abc=3abc$$

$$(a^2+b^2+c^2)^2\le\left(\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}\right)\cdot(3abc)\Rightarrow\frac{(a^2+b^2+c^2)^2}{3abc}\le\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}$$

$$a^4+b^4+c^4\ge a^2b^2+b^2c^2+c^2a^2,\ \ a^2b^2+b^2c^2+c^2a^2\ge abc(a+b+c)$$

$$a^4+b^4+c^4+2(a^2b^2+b^2c^2+c^2a^2)\ge a^2b^2+b^2c^2+c^2a^2+2abc(a+bc)\ge$$

$$\ge 3abc(a+b+c)$$

$$(a^2+b^2+c^2)^2\ge 3abc(a+b+c)$$

$$\frac{(a^2+b^2+c^2)^2}{3abc}\ge\frac{3abc(a+b+c)}{3abc}=a+b+c$$

$$a+b+c\le\frac{(a^2+b^2+c^2)^2}{3abc}\le\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}$$

$$a+b+c\le\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}\Rightarrow Q.E.D.$$

Solution 6 by Himat Mammadov-Azerbaijan

$$z_1=\sqrt{a},z_2=\sqrt{b},z_3=\sqrt{c}\ \ (z_1,z_2,z_3\in\mathbb{R}^+\subset\mathbb{C})$$

$$|z_1|^2=a,|z_2|^2=b,|z_3|^2=c$$

$$w_1=\frac{z_1^3}{z_2z_3},\ \ w_2=\frac{z_2^3}{z_3z_1},\qquad w_3=\frac{z_3^3}{z_1z_2}$$

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$$|w_1| = \left|\frac{z_1^3}{z_2 z_3}\right| = \frac{|z_1|^3}{|z_2||z_3|} = \frac{a^{\frac{3}{2}}}{\sqrt{b}\sqrt{c}} = \frac{a^{\frac{3}{2}}}{\sqrt{bc}}$$

$$|w_1|^2 = \frac{a^3}{bc},\ \ |w_2|^2 = \frac{b^3}{ca},\ \ |w_3|^2 = \frac{c^3}{ab}$$

$$|w_1|^2 + |w_2|^2 + |w_3|^2 = \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab}$$

$$w_1 w_2 = \frac{z_1^3}{z_2 z_3} \cdot \frac{z_2^3}{z_3 z_1} = \frac{z_1^2 z_2^2}{z_3^2} = \frac{a \cdot b}{c}$$

$$|w_1 w_2| = \frac{ab}{c}, |w_2 w_3| = \frac{bc}{a}, \qquad |w_3 w_1| = \frac{ca}{b}$$

$$|w_1|^2 + |w_2|^2 + \left|w_3^2\right|^2 \geq |w_1 w_2| + |w_2 w_3| + |w_3 w_1|$$

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}$$

$$\frac{ab}{c} + \frac{bc}{a} \geq 2\sqrt{\frac{ab}{c} \cdot \frac{bc}{a}} = 2b, \qquad \frac{bc}{a} + \frac{ca}{b} \geq 2c, \qquad \frac{ca}{b} + \frac{ab}{c} \geq 2a$$

$$2\left(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}\right) \geq 2(a+b+c) \Rightarrow \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \geq a+b+c$$

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \geq a+b+c$$

$$a+b+c \leq \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \Rightarrow Q.E.D.$$