ROMANIAN MATHEMATICAL MAGAZINE

JP.637 Solve for positive real numbers the following system of equations:

$$x^2+y^2=1,$$

$$1+2\cdot(3x-4x^3)\cdot(3y-4y^3)=0.$$

Proposed by Neculai Stanciu – Romania

Solution 1 by proposer

We have $1+2(3x-4x^3)(3y-4y^3)=0 \Leftrightarrow 9xy+16x^3y^3-12xy^3-12x^3y=-\frac{1}{2} \Leftrightarrow$

$$\Leftrightarrow 9xy+16(xy)^3-12xy(x^2+y^2)=-\frac{1}{2} \overset{x^2+y^2=1}{\Leftrightarrow} -3(xy)+16(xy)^3=-\frac{1}{2} \Rightarrow$$

$\Rightarrow xy\in\left\{-\frac{1}{2};\frac{1}{4}\right\} \overset{x,y>0}{\Rightarrow} xy=\frac{1}{4}$. So, $x+y=\sqrt{x^2+y^2+2xy}=\frac{\sqrt{6}}{2}$ and $x^2-\frac{\sqrt{6}}{2}x+\frac{1}{4}=0$, we get that

$$(x,y)\in\left\{\left(\frac{\sqrt{6}-\sqrt{2}}{4},\frac{\sqrt{6}+\sqrt{2}}{4}\right),\left(\frac{\sqrt{6}+\sqrt{2}}{4},\frac{\sqrt{6}-\sqrt{2}}{4}\right)\right\}.$$

Solution 2 by proposer

$$x=\cos\alpha\,,y=\sin\alpha\,;3y-4y^3=\sin 3\alpha\,,3x-4x^3=-\cos 3\alpha.$$

From $1+2(3x-4x^3)(3y-4y^3)=0$, we deduce that

$$\sin 3\alpha\cos 3\alpha=\frac{1}{2}\Rightarrow\sin 6\alpha=1\Rightarrow\alpha=\frac{1}{6}\arcsin 1=\frac{\pi}{12}.$$

Then $x+y=\cos\frac{\pi}{12}+\sin\frac{\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{4}+\frac{\sqrt{6}-\sqrt{2}}{4}=\frac{\sqrt{6}}{2}$, and as in solution 1 we obtain

$$(x,y)=\left\{\left(\frac{\sqrt{6}-\sqrt{2}}{4},\frac{\sqrt{6}+\sqrt{2}}{4}\right),\left(\frac{\sqrt{6}+\sqrt{2}}{4},\frac{\sqrt{6}-\sqrt{2}}{4}\right)\right\}$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$x^2+y^2=1,\ \ x>0,y>0\Rightarrow\ \ \exists t\in\left(0,\frac{\pi}{2}\right):\begin{cases}x=\sin t\\ y=\cos t\end{cases}$$

$$\begin{cases}3x-4x^3=3\sin t-4\sin^3 t=\sin(3t)\\ 3y-4y^3=3\cos t-4\cos^3 t=-(4\cos^3 t-3\cos t)=-\cos(3t)\end{cases}$$

$$1+2(3x-4x^3)(3y-4y^3)=0\Rightarrow 1+2\cdot\sin(3t)\cdot(-\cos(3t))=0$$

$$1-2\sin(3t)\cos(3t)=0\Rightarrow 1-\sin(2\cdot 3t)=0\Rightarrow\sin(6t)=1$$

$$t\in\left(0,\frac{\pi}{2}\right)\Rightarrow 6t\in(0,3\pi)$$

$$\sin(6t)=1, 6t\in(0,3\pi)\Rightarrow 6t\in\left\{\frac{\pi}{2},\frac{5\pi}{2}\right\}\Rightarrow\begin{cases}t_1=\dfrac{\pi}{12}\\ t_2=\dfrac{5\pi}{12}\end{cases}$$

(i) $t=\frac{\pi}{12}$

$$x=\sin\left(\frac{\pi}{12}\right)=\sin(15^\circ)=\sin(45^\circ-30^\circ)=\sin 45^\circ\cos 30^\circ-\cos 45^\circ\sin 30^\circ=$$

$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

$$y=\cos\left(\frac{\pi}{12}\right)=\cos(15^\circ)=\cos(45^\circ-30^\circ)=\cos 45^\circ\cos 30^\circ+\sin 45^\circ\sin 30^\circ=$$

$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$$

(ii) $t=\frac{5\pi}{12}$

$$x=\sin\left(\frac{5\pi}{12}\right)=\sin(75^\circ)=\sin(45^\circ+30^\circ)=\sin 45^\circ\cos 30^\circ+\cos 45^\circ\sin 30^\circ=$$

$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$$

$$y=\cos\left(\frac{5\pi}{12}\right)=\cos(75^\circ)=\cos(45^\circ+30^\circ)=\cos 45^\circ\cos 30^\circ-\sin 45^\circ\sin 30^\circ=$$

$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

$$(x,y)\in\left\{\left(\frac{\sqrt{6}-\sqrt{2}}{4},\frac{\sqrt{6}+\sqrt{2}}{4}\right),\left(\frac{\sqrt{6}+\sqrt{2}}{4},\frac{\sqrt{6}-\sqrt{2}}{4}\right)\right\}$$

Solution 4 by Marin Circiu-Romania

We make the substitution $(x,y)=(\sin t,\cos t), t\in\left(0,\frac{\pi}{2}\right)$ and

$\sin 3t=3\sin t-4\sin^3 t,$

$\cos 3t=4\cos^3 t-3\cos t\Rightarrow \sin 3t=3x-4x^3, \cos 3t=4y^3-3y.$

We obtain $1-2\sin 3t\cos 3t=0\Leftrightarrow 1-\sin 6t=0\Leftrightarrow \sin 6t=1\Leftrightarrow 6t=\frac{\pi}{2}+2k\pi$

$k=0,1\Leftrightarrow t=\frac{\pi}{12}+\frac{k\pi}{3}, k=0,1.$

1. $k=0\Rightarrow t=\frac{\pi}{12}\Rightarrow(x,y)=\left(\sin\frac{\pi}{12},\cos\frac{\pi}{12}\right)\Leftrightarrow(x,y)=\left(\frac{\sqrt{6}-\sqrt{2}}{4},\frac{\sqrt{6}+\sqrt{2}}{4}\right).$

1. $k=1\Rightarrow t=\frac{\pi}{12}\Rightarrow(x,y)=\left(\sin\frac{5\pi}{12},\cos\frac{5\pi}{12}\right)\Leftrightarrow(x,y)=\left(\frac{\sqrt{6}+\sqrt{2}}{4},\frac{\sqrt{6}-\sqrt{2}}{4}\right)$

The set of the solutions of the system is $S=\left\{\left(\frac{\sqrt{6}-\sqrt{2}}{4},\frac{\sqrt{6}+\sqrt{2}}{4}\right),\left(\frac{\sqrt{6}+\sqrt{2}}{4},\frac{\sqrt{6}-\sqrt{2}}{4}\right)\right\}$.

Solution 5 by Jose Luis Diaz Barrero-Spain

Since $x,y>0$ and $x^2+y^2=1$, we can write $x=\sin\alpha$, $y=\cos\alpha$, $0<\alpha<\frac{\pi}{2}$. The triple-angle formulas give

$$3x-4x^3=3\sin\alpha-4\sin^3\alpha=\sin 3\alpha,$$

$$3y-4y^3=3\cos\alpha-4\cos^3\alpha=-\cos 3\alpha.$$

Substituting into the second equation, we get

$$1-2\sin 3\alpha\cos 3\alpha=0\Leftrightarrow 1-\sin 6\alpha=0\Leftrightarrow\sin 6\alpha=1.$$

Hence $6\alpha=\frac{\pi}{2}+2k\pi$, that is, $\alpha=\frac{\pi}{12}+\frac{k\pi}{3}$ with $k\in\mathbb{Z}$. Since $\alpha\in\left(0,\frac{\pi}{2}\right)$, the only admissible values are

$$\alpha=\frac{\pi}{12}\text{ or }\alpha=\frac{5\pi}{12}.$$

Using $\sin\frac{\pi}{12}=\cos\frac{5\pi}{12}=\frac{\sqrt{6}-\sqrt{2}}{4}$ and $\cos\frac{\pi}{12}=\sin\frac{5\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{4}$, the solutions of the system are

$$(x,y)=\left(\frac{\sqrt{6}-\sqrt{2}}{4},\frac{\sqrt{6}+\sqrt{2}}{4}\right)\quad\text{and}\quad(x,y)=\left(\frac{\sqrt{6}+\sqrt{2}}{4},\frac{\sqrt{6}-\sqrt{2}}{4}\right).$$