ROMANIAN MATHEMATICAL MAGAZINE

JP.636 If $a, b > 0$ and $ab = 1$ then:

$$\frac{a}{\sqrt{a^2+1}} + \frac{b}{\sqrt{b^2+1}} \leq \sqrt{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$ab = 1 \Rightarrow b = \frac{1}{a}$$

$$\frac{a}{\sqrt{a^2+1}} + \frac{b}{\sqrt{b^2+1}} = \frac{a}{\sqrt{a^2+1}} + \frac{\frac{1}{a}}{\sqrt{\frac{1}{a^2}+1}} = \frac{a}{\sqrt{a^2+1}} + \frac{1}{\sqrt{a^2+1}} = \frac{a+1}{\sqrt{a^2+1}} \leq \sqrt{2} \quad \text{(to prove)}$$

$$a+1 \leq \sqrt{2}\cdot\sqrt{a^2+1}, \qquad (a+1)^2 \leq 2(a^2+1), \qquad a^2+2a+1 \leq 2a^2+2$$

$$2a^2+2-a^2-2a-1 \geq 0,\ a^2-2a+1 \geq 0,\ (a-1)^2 \geq 0 \quad \text{(True!)}$$

Equality holds for $a = b = 1$.

Solution 2 by Marin Chirciu– Romania

$$\frac{a}{\sqrt{a^2+2}} + \frac{b}{\sqrt{b^2+1}} \leq \sqrt{2} \Leftrightarrow a\sqrt{b^2+1} + b\sqrt{a^2+1} \leq \sqrt{2}\sqrt{a^2+1}\sqrt{b^2+1} \Leftrightarrow$$

$$\Leftrightarrow \sqrt{a^2b^2+a^2} + \sqrt{a^2b^2+b^2} \leq \sqrt{2}\sqrt{a^2b^2+a^2+b^2+1}, ab = 1 \Leftrightarrow$$

$$\sqrt{1+a^2} + \sqrt{1+b^2} \leq \sqrt{2}\sqrt{2+a^2+b^2} \Leftrightarrow \left(\sqrt{1+a^2} + \sqrt{1+b^2}\right)^2 \leq 2(2+a^2+b^2) \Leftrightarrow$$

$$\Leftrightarrow (1+a^2) + (1+b^2) + 2\sqrt{1+a^2}\sqrt{1+b^2} \leq 2(2+a^2+b^2) \Leftrightarrow$$

$$\Leftrightarrow 2\sqrt{1+a^2}\sqrt{1+b^2} \leq 2+a^2+b^2, ab = 1 \Leftrightarrow 2\sqrt{2+a^2+b^2} \leq 2+a^2+b^2 \Leftrightarrow$$

$$\Leftrightarrow 2 \leq \sqrt{2+a^2+b^2} \Leftrightarrow 4 \leq 2+a^2+b^2 \Leftrightarrow a^2+b^2 \geq 2, \text{ see } a^2+b^2 \geq 2ab = 2\cdot 1 = 2$$

Equality holds if and only if $a = b = 1$. Remark:

If $a, b > 0, ab = 1$ and $0 \leq \lambda \leq \frac{5}{2}$ then:

$$\frac{a}{\sqrt{a^2+\lambda}} + \frac{b}{\sqrt{b^2+\lambda}} \leq \frac{2}{\sqrt{\lambda+1}}$$

Marin Chirciu

Solution

$$\frac{a}{\sqrt{a^2+\lambda}}+\frac{b}{\sqrt{b^2+\lambda}}\le\frac{2}{\sqrt{\lambda+1}}\Leftrightarrow\sqrt{\lambda+1}\left(a\sqrt{b^2+\lambda}+b\sqrt{a^2+\lambda}\right)\le 2\sqrt{a^2+\lambda}\sqrt{b^2+\lambda}$$

$$\Leftrightarrow\sqrt{\lambda+1}\left(\sqrt{a^2b^2+\lambda a^2}+\sqrt{a^2b^2+\lambda b^2}\right)\le 2\sqrt{a^2b^2+\lambda a^2+\lambda b^2+\lambda^2},ab=1\Leftrightarrow$$

$$\Leftrightarrow\sqrt{\lambda+1}\left(\sqrt{1+\lambda a^2}+\sqrt{1+\lambda b^2}\right)\le 2\sqrt{1+\lambda a^2+\lambda b^2+\lambda^2},ab=1\Leftrightarrow$$

$$\Leftrightarrow(\lambda+1)\left(\sqrt{1+\lambda a^2}+\sqrt{1+\lambda b^2}\right)^2\le 4(1+\lambda a^2+\lambda b^2+\lambda^2),ab=1\Leftrightarrow$$

$$\Leftrightarrow(\lambda+1)\left((1+\lambda a^2)+(1+\lambda b^2)+2\sqrt{1+\lambda a^2}\sqrt{1+\lambda b^2}\right)\le 4(1+\lambda a^2+\lambda b^2+\lambda^2)$$

$$\Leftrightarrow(\lambda+1)\left(2+\lambda a^2+\lambda b^2+2\sqrt{1+\lambda a^2+\lambda b^2+\lambda^2}\right)\le 4(1+\lambda a^2+\lambda b^2+\lambda^2).$$

Denoting $t=\sqrt{1+\lambda a^2+\lambda b^2+\lambda^2}$, the inequality can be written:

$$(\lambda+1)(1+t^2-\lambda^2+2t)\le 4t^2\Leftrightarrow(3-\lambda)t^2-2(\lambda+1)t+(\lambda+1)^2(\lambda-1)\ge 0\Leftrightarrow$$

$$\Leftrightarrow(t-\lambda-1)\left(t-\tfrac{(\lambda+1)(\lambda-2)}{3-\lambda}\right)\ge 0,\text{ see } t\ge\lambda+1\ge\tfrac{(\lambda+1)(\lambda-2)}{3-\lambda},0\le\lambda\le\tfrac{5}{2}.$$

We have used $t\ge\lambda+1\Leftrightarrow t=\sqrt{1+\lambda a^2+\lambda b^2+\lambda^2}\ge\lambda+1\Leftrightarrow$

$$\Leftrightarrow 1+\lambda a^2+\lambda b^2+\lambda^2\ge(\lambda+1)^2\Leftrightarrow$$

$$\Leftrightarrow a^2+b^2\ge 2,\text{ see } a^2+b^2\ge 2ab=2\cdot 1=2.$$

Equality holds if and only if $a=b=1$. Note: For $\lambda=1$ we obtain Problem JP.636 from RMM 43, Winter 2026.

Solution 3 by Jose Luis Diaz Barrero-Spain

On account of the inequality $(x+y)^2\le 2(x^2+y^2)$, we have

$$\left(\frac{a}{\sqrt{a^2+1}}+\frac{b}{\sqrt{b^2+1}}\right)^2\le 2\left(\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}\right)$$

Since $ab=1$, we get $b=\frac{1}{a}$, and so

$$\frac{b^2}{b^2+1}=\frac{\frac{1}{a^2}}{\frac{1}{a^2}+1}=\frac{1}{1+a^2}$$

Hence

$$\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}=\frac{a^2}{a^2+1}+\frac{1}{a^2+1}=1,$$

and the square of the left-hand side is at most 2. Because the left-hand side is positive, it is at most $\sqrt{2}$. Equality holds if and only if $a = b = 1$.

Alternative solution. Since $a > 0$, there exists a unique $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that $a = \tan\alpha$. From $ab = 1$ we get $b = \frac{1}{\tan\alpha} = \cot\alpha$. As $\sin\alpha > 0$ and $\cos\alpha > 0$,

$$\sqrt{a^2+1} = \sqrt{\tan^2\alpha + 1} = \frac{1}{\cos\alpha}, \quad \sqrt{b^2+1} = \sqrt{\cot^2\alpha + 1} = \frac{1}{\sin\alpha}.$$

Therefore

$$\frac{a}{\sqrt{a^2+1}} = \tan\alpha\cos\alpha = \sin\alpha, \qquad \frac{b}{\sqrt{b^2+1}} = \cot\alpha\sin\alpha = \cos\alpha$$

and the left-hand side of the inequality equals

$$\sin\alpha + \cos\alpha = \sqrt{2}\left(\frac{\sqrt{2}}{2}\sin\alpha + \frac{\sqrt{2}}{2}\cos\alpha\right) = \sqrt{2}\sin\left(\alpha + \frac{\pi}{4}\right) \le \sqrt{2}.$$

Equality holds if and only if $\alpha + \frac{\pi}{4} = \frac{\pi}{2}$, that is, $\alpha = \frac{\pi}{4}$, which corresponds to $a = b = 1$.